MINISTRY OF SCIENCE AND HIGHER EDUCATION

Mathematics for Natural Sciences

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Chapter One
Propositional Logic and Set Theory

In this chapter, we study the basic concepts of propositional logic and some part of set theory. In the first part, we deal about propositional logic, logical connectives, quantifiers and arguments. In the second part, we turn our attention to set theory and discus about description of sets and operations of sets.

Main Objectives of this Chapter

At the end of this chapter, students will be able to:-

- ❖ Know the basic concepts of mathematical logic.
- ❖ Know methods and procedures in combining the validity of statements.
- ❖ Understand the concept of quantifiers.
- ❖ Know basic facts about argument and validity.
- ❖ Understand the concept of set.
- ❖ Apply rules of operations on sets to find the result.
- ❖ Show set operations using Venn diagrams.

1.1. Propositional Logic

Mathematical or symbolic logic is an analytical theory of the art of reasoning whose goal is to systematize and codify principles of valid reasoning. It has emerged from a study of the use of language in argument and persuasion and is based on the identification and examination of those parts of language which are essential for these purposes. It is formal in the sense that it lacks reference to meaning. Thereby it achieves versatility: it may be used to judge the correctness of a chain of reasoning (in particular, a "mathematical proof") solely on the basis of the form (and not the content) of the sequence of statements which make up the chain. There is a variety of symbolic logics. We shall be concerned only with that one which encompasses most of the deductions of the sort encountered in mathematics. Within the context of logic itself, this is "classical" symbolic logic.

Section objectives:

After completing this section, students will be able to:-

- ✓ Identify the difference between proposition and sentence.
- ✓ Describe the five logical connectives.
- ✓ Determine the truth values of propositions using the rules of logical connectives.

- ✓ Construct compound propositions using the five logical connectives.
- ✓ Identify the difference between the converse and contrapositive of conditional statements.
- ✓ Determine the truth values of compound propositions.
- ✓ Distinguish a given compound proposition is whether tautology or contradiction.

1.1.1. Definition and examples of propositions

Consider the following sentences.

a. 2 is an even number.
b. A triangle has four sides.
c. Athlete Haile G/silassie weighed 45 kg when he was 20 years old.
d. May God bless you!
e. Give me that book.
f. What is your name?

The first three sentences are declarative sentences. The first one is true and the second one is false. The truth value of the third sentence cannot be ascertained because of lack of historical records but it is, by its very form, either true or false but not both. On the other hand, the last three sentences have no truth value. So they are not declaratives.

Now we begin by examining proposition, the building blocks of every argument. A proposition is a sentence that may be asserted or denied. Proposition in this way are different from questions, commands, and exclamations. Neither questions, which can be asked, nor exclamations, which can be uttered, can possibly be asserted or denied. Only propositions assert that something is (or is not) the case, and therefore only they can be true or false.

Definition 1.1: A proposition (or statement) is a sentence which has a truth value (either True or False but not both).

The above definition does not mean that we must always know what the truth value is. For example, the sentence "The 1000^{th} digit in the decimal expansion of π is 7" is a proposition, but it may be necessary to find this information in a Web site on the Internet to determine whether this statement is true. Indeed, for a sentence to be a proposition (or a statement), it is not a requirement that we are able to determine its truth value.

Every proposition has a truth value, namely **true** (denoted by ***T***) or **false** (denoted by ***F***).

1.1.2. Logical connectives

In mathematical discourse and elsewhere one constantly encounters declarative sentences which have been formed by modifying a statement with the word "not" or by connecting statements with the words "and", "or", "if . . . then (or implies)", and "if and only if". These five words or combinations of words are called propositional connectives.

Note: Letters such as p, q, r, s etc. are usually used to denote propositions.

Conjunction

When two propositions are joined with the connective "**and**," the proposition formed is a logical ***conjunction***. "and" is denoted by "$\wedge$". So, the logical conjunction of two propositions, p and q, is written:

$p \wedge q$, read as "p and q," or "p conjunction q".

p and q are called ***the components of the conjunction***. $p \wedge q$ is true if and only if p is true and q is true.

The truth table for conjunction is given as follows:

p	q	$p \wedge q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***F***
F	***F***	***F***

Example 1.1: Consider the following propositions:

p: 3 is an odd number. (True)

q: 27 is a prime number. (False)

r: Addis Ababa is the capital city of Ethiopia. (True)

a. $p \wedge q$: 3 is an odd number and 27 is a prime number. (False)
b. $p \wedge r$: 3 is an odd number and Addis Ababa is the capital city of Ethiopia. (True)

Disjunction

When two propositions are joined with the connective "**or**," the proposition formed is called a logical ***disjunction***. "or" is denoted by "$\vee$". So, the logical disjunction of two propositions, p and q, is written:

$p \vee q$ read as "p or q" or "p disjunction q."

$p \vee q$ is false if and only if both p and q are false.

The truth table for disjunction is given as follows:

p	q	$p \vee q$
T	***T***	***T***
T	***F***	***T***
F	***T***	***T***
F	***F***	***F***

Example 1.2: Consider the following propositions:

p: 3 is an odd number. (True)

q: 27 is a prime number. (False)

s: Nairobi is the capital city of Ethiopia. (False)

a. $p \vee q$: 3 is an odd number or 27 is a prime number. (True)
b. p ∨ r: 27 is a prime number or Nairobi is the capital city of Ethiopia. (False)

Note: The use of "**or**" in propositional logic is rather different from its normal use in the English language. For example, if Solomon says, "I will go to the football match in the afternoon or I will go to the cinema in the afternoon," he means he will do one thing or the other, but not both. Here "or" is used in the exclusive sense. But in propositional logic, "or" is used in the inclusive sense; that is, we allow Solomon the possibility of doing both things without him being inconsistent.

Implication

When two propositions are joined with the connective "**implies**," the proposition formed is called a *logical* ***implication***. "implies" is denoted by "⟹." So, the logical implication of two propositions, p and q, is written:

$p \Longrightarrow q$ read as "p implies q."

The function of the connective "implies" between two propositions is the same as the use of "If … then …" Thus $p \Longrightarrow q$ can be read as "if p, then q."

$p \Longrightarrow q$ is false if and only if p is true and q is false.

This form of a proposition is common in mathematics. The proposition p is called the hypothesis or the antecedent of the conditional proposition $p \Longrightarrow q$ while q is called its conclusion or the consequent.

The following is the truth table for implication.

p	q	$p \Longrightarrow q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***T***
F	***F***	***T***

Examples 1.3: Consider the following propositions:

p: 3 is an odd number. (True)

q: 27 is a prime number. (False)

r: Addis Ababa is the capital city of Ethiopia. (True)

$p \Longrightarrow q$: If 3 is an odd number, then 27 is prime. (False)

$p \Longrightarrow r$: If 3 is an odd number, then Addis Ababa is the capital city of Ethiopia. (True)

We have already mentioned that $p \Longrightarrow q$ can be expressed as both "If p, then q" and "p implies q." There are various ways of expressing the proposition $p \Longrightarrow q$, namely:

If p, then q.
q if p.
p implies q.
p only if q.
p is sufficient for q.
q is necessary for p

Bi-implication

When two propositions are joined with the connective "**bi-implication**," the proposition formed is called a *logical* ***bi-implication*** or a *logical* ***equivalence***. A bi-implication is denoted by "$\Leftrightarrow$". So the logical bi-implication of two propositions, p and q, is written:

$$p \Leftrightarrow q.$$

$p \Leftrightarrow q$ is false if and only if p and q have different truth values.

The truth table for bi-implication is given by:

p	q	$p \Leftrightarrow q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***F***
F	***F***	***T***

Examples 1.4:

a. Let p: 2 is greater than 3. (False)
   q: 5 is greater than 4. (True)

Then

$p \Leftrightarrow q$: 2 is greater than 3 if and only if 5 is greater than 4. (False)

b. Consider the following propositions:
   p: 3 is an odd number. (True)
   q: 2 is a prime number. (True)
   $p \Leftrightarrow q$: 3 is an odd number if and only if 2 is a prime number. (True)

There are various ways of stating the proposition $p \Leftrightarrow q$.

p if and only if q (also written as p iff q),
p implies q and q implies p,
p is necessary and sufficient for q
q is necessary and sufficient for p
p is equivalent to q

Negation

Given any proposition p, we can form the proposition $\neg p$ called the **negation** of p. The truth value of $\neg p$ is F if p is T and T if p is F.

We can describe the relation between p and $\neg p$ as follows.

p	$\neg p$
T	**F**
F	**T**

Example 1.5: Let p: Addis Ababa is the capital city of Ethiopia. (True)

$\neg p$: Addis Ababa is not the capital city of Ethiopia. (False)

Exercises

1. Which of the following sentences are propositions? For those that are, indicate the truth value.
   a. 123 is a prime number.
   b. 0 is an even number.
   c. $x^2 - 4 = 0$.
   d. Multiply $5x + 2$ by 3.
   e. What an impossible question!
2. State the negation of each of the following statements.
   a. $\sqrt{2}$ is a rational number.
   b. 0 is not a negative integer.
   c. 111 is a prime number.
3. Let p: 15 is an odd number.
   q: 21 is a prime number.
   State each of the following in words, and determine the truth value of each.
   a. $p \vee q$.
   b. $p \wedge q$.
   c. $\neg p \vee q$.
   d. $p \wedge \neg q$.
   e. $p \Longrightarrow q$.
   f. $q \Longrightarrow p$.
   a. $\neg p \Longrightarrow \neg q$.
   g. $\neg q \Longrightarrow \neg p$.

4. Complete the following truth table.

p	q	$\neg q$	$p \wedge \neg q$
T	T		
T	F		
F	T		
F	F		

1.1.3. Compound (or complex) propositions

So far, what we have done is simply to define the logical connectives, and express them through algebraic symbols. Now we shall learn how to form propositions involving more than one connective, and how to determine the truth values of such propositions.

Definition 1.2: The proposition formed by joining two or more proposition by connective(s) is called a compound statement.

Note: We must be careful to insert the brackets in proper places, just as we do in arithmetic. For example, the expression $p \Longrightarrow q \wedge r$ will be meaningless unless we know which connective should apply first. It could mean $(p \Longrightarrow q) \wedge r$ or $p \Longrightarrow (q \wedge r)$, which are very different propositions. The truth value of such complicated propositions is determined by systematic applications of the rules for the connectives.

The possible truth values of a proposition are often listed in a table, called a **truth table**. If p and q are propositions, then there are four possible combinations of truth values for p and q. That is, TT, TF, FT and FF. If a third proposition r is involved, then there are eight possible combinations of truth values for p,q and r. In general, a truth table involving "n" propositions $p_1, p_2,\ldots, p_n$ contains 2^n possible combinations of truth values. So, we use truth tables to determine the truth value of a compound proposition based on the truth value of its constituent component propositions.

Examples 1.6:

a. Suppose p and r are true and q and s are false.
   What is the truth value of $(p \wedge q) \Longrightarrow (r \vee s)$?
   i. Since p is true and q is false, $p \wedge q$ is false.
   ii. Since r is true and s is false, $r \vee s$ is true.
   iii. Thus by applying the rule of implication, we get that $(p \wedge q) \Longrightarrow (r \vee s)$ is true.
b. Suppose that a compound proposition is symbolized by

$$(p \vee q) \Longrightarrow (r \Leftrightarrow \neg s)$$

and that the truth values of p, q, r, and s are T, F, F, and T, respectively. Then the truth value of $p \vee q$ is T, that of $\neg s$ is F, that of $r \Leftrightarrow \neg s$ is T. So the truth value of $(p \vee q) \Rightarrow (r \Leftrightarrow \neg s)$ is T.

Remark: When dealing with compound propositions, we shall adopt the following convention on the use of parenthesis. Whenever "$\vee$" or "$\wedge$" occur with "$\Rightarrow$" or "$\Leftrightarrow$", we shall assume that "$\vee$" or "$\wedge$" is applied first, and then "$\Rightarrow$" or "$\Leftrightarrow$" is then applied. For example,

$$p \wedge q \Rightarrow r \text{ means } (p \wedge q) \Rightarrow r$$

$$p \vee q \Leftrightarrow r \text{ means } (p \vee q) \Leftrightarrow r$$

$$\neg q \Rightarrow \neg p \text{ means } (\neg q) \Rightarrow (\neg p)$$

$$\neg q \Rightarrow p \Leftrightarrow r \text{ means } ((\neg q) \Rightarrow p) \Leftrightarrow r$$

However, it is always advisable to use brackets to indicate the order of the desired operations. .

Definition 1.3: Two compound propositions P and Q are said to be ***equivalent*** if they have the same truth value for all possible combinations of truth values for the component propositions occurring in both P and Q. In this case we write $P \equiv Q$.

Example 1.7: Let $P: p \Rightarrow q$.
Q: ¬(p ∧ ¬q)

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	***T***	***F***	***F***	***T***	***T***
T	***F***	***F***	***T***	***F***	***F***
F	***T***	***T***	***F***	***T***	***T***
F	***F***	***T***	***T***	***T***	***T***

Then, P is equivalent to Q, since columns 5 and 6 of the above table are identical.

Example 1.8: Let $P: p \Rightarrow q$.
$Q: \neg p \Rightarrow \neg q$.
Then

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$
T	***T***	***F***	***F***	***T***	***T***
T	***F***	***F***	***T***	***F***	***T***
F	***T***	***T***	***F***	***T***	***F***
F	***F***	***T***	***T***	***T***	***T***

Looking at columns 5 and 6 of the table we see that they are not identical. Thus $P \not\equiv Q$.

It is useful at this point to mention the non-equivalence of certain conditional propositions. Given the conditional $p \Rightarrow q$, we give the related conditional propositions:-

$q \Rightarrow p$: Converse of $p \Rightarrow q$

$\neg p \Rightarrow \neg q$: Inverse of $p \Rightarrow q$

$\neg q \Rightarrow \neg p$: Contrapositive of $p \Rightarrow q$

As we observed from example 1.7, the conditional $p \Rightarrow q$ and its contrapositve $\neg q \Rightarrow \neg p$ are equivalent. On the other hand, $p \Rightarrow q \not\equiv q \Rightarrow p$ and $p \Rightarrow q \not\equiv \neg q \Rightarrow \neg p$.

Do not confuse the contrapositive and the converse of the conditional proposition. Here is the difference:

Converse: The hypothesis of a converse statement is the conclusion of the conditional statement and the conclusion of the converse statement is the hypothesis of the conditional statement.

Contrapositive: The hypothesis of a contrapositive statement is the negation of conclusion of the conditional statement and the conclusion of the contrapositive statement is the negation of hypothesis of the conditional statement.

Example 1.9:

a. If Kidist lives in Addis Ababa, then she lives in Ethiopia.
   Converse: If Kidist lives in Ethiopia, then she lives in Addis Ababa.
   Contrapositive: If Kidist does not live in Ethiopia, then she does not live in Addis Ababa.
   Inverse: If Kidist does not live in Addis Ababa, then she does not live in Ethiopia.
b. If it is morning, then the sun is in the east.
   Converse: If the sun is in the east, then it is morning.
   Contrapositive: If the sun is not in the east, then it is not morning.
   Inverse: If it is not morning, then the sun is not the east.

Propositions, under the relation of logical equivalence, satisfy various laws or identities, which are listed below.

1. Idempotent Laws
   a. $p \equiv p \vee p$.
   b. $p \equiv p \wedge p$.
2. Commutative Laws
   a. $p \wedge q \equiv q \wedge p$.
   b. $p \vee q \equiv q \vee p$.
3. Associative Laws
   a. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.
   b. $p \vee (q \vee r) \equiv (p \vee q) \vee r$.
4. Distributive Laws
   a. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
   b. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
5. De Morgan's Laws
   a. $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

b. $\neg(p \vee q) \equiv \neg p \wedge \neg q$

6. Law of Contrapositive

$p \Longrightarrow q \equiv \neg q \Longrightarrow \neg p$

7. Complement Law

$\neg(\neg p) \equiv p$.

1.1.4. Tautology and contradiction

Definition: A compound proposition is a **tautology** if it is always true regardless of the truth values of its component propositions. If, on the other hand, a compound proposition is always false regardless of its component propositions, we say that such a proposition is a **contradiction**.

Note: A proposition that is neither a **tautology** nor a **contradiction** is called a **contingency.**

Examples 1.10:

a. Suppose p is any proposition. Consider the compound propositions $p \vee \neg p$ and $p \wedge \neg p$.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	**F**	***T***	***F***
F	**T**	***T***	***F***

Observe that $p \vee \neg p$ is a tautology while $p \wedge \neg p$ is a contradiction.

b. For any propositions p and q. Consider the compound proposition $p \Longrightarrow (q \Longrightarrow p)$. Let us make a truth table and study the situation.

p	q	$q \Longrightarrow p$	$p \Longrightarrow (q \Longrightarrow p)$
T	***T***	***T***	**T**
T	***F***	***T***	**T**
F	***T***	***F***	**T**
F	***F***	***T***	**T**

We have exhibited all the possibilities and we see that for all truth values of the constituent propositions, the proposition $p \Longrightarrow (q \Longrightarrow p)$ is always true. Thus, $p \Longrightarrow (q \Longrightarrow p)$ is a tautology.

c. The truth table for the compound proposition $(p \Longrightarrow q) \Leftrightarrow (p \wedge \neg q)$.

p	q	$\neg q$	$p \wedge \neg q$	$p \Longrightarrow q$	$(p \Longrightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	***T***	***F***	***F***	***T***	***F***
T	***F***	***T***	***T***	***F***	***F***
F	***T***	***F***	***F***	***T***	***F***
F	***F***	***T***	***F***	***T***	***F***

In example 1.10(c), the given compound proposition has a truth value F for every possible combination of assignments of truth values for the component propositions p and q. Thus $(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$ is a contradiction.

Remark:

1. In a truth table, if a proposition is a tautology, then every line in its column has T as its entry; if a proposition is a contradiction, every line in its column has F as its entry.
2. Two compound propositions P and Q are equivalent if and only if "$P \Leftrightarrow Q$" is a tautology.

Exercises

1. For statements p, q and r, use a truth table to show that each of the following pairs of statements are logically equivalent.
   a. $(p \wedge q) \Leftrightarrow p$ and $p \Rightarrow q$.
   b. $p \Rightarrow (q \vee r)$ and $\neg q \Rightarrow (\neg p \vee r)$.
   c. $(p \vee q) \Rightarrow r$ and $(p \Rightarrow q) \wedge (q \Rightarrow r)$.
   d. $p \Rightarrow (q \vee r)$ and $(\neg r) \Rightarrow (p \Rightarrow q)$.
   e. $p \Rightarrow (q \vee r)$ and $((\neg r) \wedge p) \Rightarrow q$.
2. For statements p, q, and r, show that the following compound statements are tautology.
   a. $p \Rightarrow (p \vee q)$.
   b. $(p \wedge (p \Rightarrow q)) \Rightarrow q$.
   c. $\big((p \Rightarrow q) \wedge (q \Rightarrow r)\big) \Rightarrow (p \Rightarrow r)$.
3. For statements p and q, show that $(p \wedge \neg q) \wedge (p \wedge q)$ is a contradiction.
4. Write the contrapositive and the converse of the following conditional statements.
   a. If it is cold, then the lake is frozen.
   b. If Solomon is healthy, then he is happy.
   c. If it rains, Tigist does not take a walk.
5. Let p and q be statements. Which of the following implies that $p \vee q$ is false?
   a. $\neg p \vee \neg q$ is false.
   b. $\neg p \vee q$ is true.
   c. $\neg p \wedge \neg q$ is true.
   d. $p \Rightarrow q$ is true.
   e. $p \wedge q$ is false.
6. Suppose that the statements $p, q, r,$ and s are assigned the truth values $T, F, F,$ and T, respectively. Find the truth value of each of the following statements.
   a. $(p \vee q) \vee r$.
   b. $p \vee (q \vee r)$.
   c. $r \Rightarrow (s \wedge p)$.
   d. $p \Rightarrow (r \Rightarrow s)$.
   e. $p \Rightarrow (r \vee s)$.
   f. $(p \vee r) \Leftrightarrow (r \wedge \neg s)$.
   g. $(s \Leftrightarrow p) \Rightarrow (\neg p \vee s)$.
   h. $(q \wedge \neg s) \Rightarrow (p \Leftrightarrow s)$.
   i. $(r \wedge s) \Rightarrow (p \Rightarrow (\neg q \vee s))$.
   j. $(p \vee \neg q) \vee r \Rightarrow (s \wedge \neg s)$.

7. Suppose the value of $p \Rightarrow q$ is T; what can be said about the value of $\neg p \land q \Leftrightarrow p \lor q$?
8. a. Suppose the value of $p \Leftrightarrow q$ is T; what can be said about the values of $p \Leftrightarrow \neg q$ and $\neg p \Leftrightarrow q$?

   b. Suppose the value of $p \Leftrightarrow q$ is F; what can be said about the values of $p \Leftrightarrow \neg q$ and $\neg p \Leftrightarrow q$?
9. Construct the truth table for each of the following statements.
   a. $p \Rightarrow (p \Rightarrow q)$.
   b. $(p \lor q) \Leftrightarrow (q \lor p)$.
   c. $p \Rightarrow \neg(q \land r)$.
   d. $(p \Rightarrow q) \Leftrightarrow (\neg p \lor q)$.
   e. $\big(p \Rightarrow (q \land r)\big) \lor (\neg p \land q)$.
   f. $(p \land q) \Rightarrow ((q \land \neg q) \Rightarrow (r \land q))$.
10. For each of the following determine whether the information given is sufficient to decide the truth value of the statement. If the information is enough, state the truth value. If it is insufficient, show that both truth values are possible.
    a. $(p \Rightarrow q) \Rightarrow r$, where $r = T$.
    b. $p \land (q \Rightarrow r)$, where $q \Rightarrow r = T$.
    c. $p \lor (q \Rightarrow r)$, where $q \Rightarrow r = T$.
    d. $\neg(p \lor q) \Leftrightarrow (\neg p \land \neg q)$, where $p \lor q = T$.
    e. $(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$, where $q = T$.
    f. $(p \land q) \Rightarrow (p \lor s)$, where $p = T$ and $s = F$.

1.2. Open propositions and quantifiers

In mathematics, one frequently comes across sentences that involve a variable. For example, $x^2 + 2x - 3 = 0$ is one such. The truth value of this statement depends on the value we assign for the variable x. For example, if $x = 1$, then this sentence is true, whereas if $x = -1$, then the sentence is false.

Section objectives:

After completing this section, students will be able to:-

- ✓ Define open proposition.
- ✓ Explain and exemplify the difference between proposition and open proposition.
- ✓ Identify the two types of quantifiers.
- ✓ Convert open propositions into propositions using quantifiers.
- ✓ Determine the truth value of a quantified proposition.
- ✓ Convert a quantified proposition into words and vise versa.
- ✓ Explain the relationship between existential and universal quantifiers.

- ✓ Analyze quantifiers occurring in combinations.

Definition 1.4: An open statement (also called a predicate) is a sentence that contains one or more variables and whose truth value depends on the values assigned for the variables. We represent an open statement by a capital letter followed by the variable(s) in parenthesis, e.g., $P(x), Q(x), R(x, y)$ etc.

Example 1.11: Here are some open propositions:

a. x is the day before Sunday.
b. y is a city in Africa.
c. x is greater than y.
d. $x + 4 = -9$.

It is clear that each one of these examples involves variables, but is not a proposition as we cannot assign a truth value to it. However, if individuals are substituted for the variables, then each one of them is a proposition or statement. For example, we may have the following.

a. Monday is the day before Sunday.
b. London is a city in Africa.
c. 5 is greater than 9.
d. –13 + 4= –9

Remark

The collection of all allowable values for the variable in an open sentence is called the **universal set** (the universe of discourse) and denoted by $\boldsymbol{U}$.

Definition 1.5: Two open proposition $P(x)$ and $Q(x)$ are said to be equivalent if and only if

$P(a) = Q(a)$ for all individual a. Note that if the universe U is specified, then $P(x)$ and $Q(x)$ are equivalent if and only if $P(a) = Q(a)$ for all $a \in U$.

Example 1.12: Let $P(x): x^2 - 1 = 0$.

$Q(x): |x| \geq 1$.

Let $U = \{-1, -\frac{1}{2}, 0,1\}$.

Then for all $a \in U$; $P(a)$ and $Q(a)$ have the same truth value.

$P(-1): (-1)^2 - 1 = 0 \quad (T) \qquad Q(-1): |-1| \geq 1 \quad (T)$

$P\left(-\frac{1}{2}\right): (-\frac{1}{2})^2 - 1 = 0 \quad (F) \qquad Q\left(-\frac{1}{2}\right): \left|-\frac{1}{2}\right| \geq 1 \quad (F)$

$P(0): 0 - 1 = 0 \quad (F) \qquad Q(0): |0| \geq 1 \quad (F)$

$P(1): 1 - 1 = 0 \quad (T) \qquad Q(1): |1| \geq 1 \quad (T)$

Therefore $P(a) = Q(a)$ for all $a \in U$.

Definition 1.6: Let U be the universal set. An open proposition $P(x)$ is a tautology if and only if $P(a)$ is always true for all values of $a \in U$.

Example 1.13: The open proposition $P(x): x^2 \geq 0$ is a tautology.

As we have observed in example 1.11, an open proposition can be converted into a proposition by substituting the individuals for the variables. However, there are other ways that an open proposition can be converted into a proposition, namely by a method called quantification. Let $P(x)$ be an open proposition over the domain S. Adding the phrase "For every $x \in S$" to $P(x)$ or "For some $x \in S$" to $P(x)$ produces a statement called a quantified statement.

Consider the following open propositions with universe $\mathbb{R}$.

a. $R(x): x^2 \geq 0$.
b. $P(x): (x+2)(x-3) = 0$.
c. $Q(x): x^2 < 0$.

Then $R(x)$ is always true for each $x \in \mathbb{R}$,
$P(x)$ is true only for $x = -2$ and $x = 3$,
$Q(x)$ is always false for all values of $x \in \mathbb{R}$.

Hence, given an open proposition $P(x)$, with universe U, we observe that there are three possibilities.

a. $P(x)$ is true for all $x \in U$.
b. $P(x)$ is true for some $x \in U$.
c. $P(x)$ is false for all $x \in U$.

Now we proceed to study open propositions which are satisfied by "**all**" and "**some**" members of the given universe.

a. The phrase "for every x " is called a **universal quantifier**. We regard "for every x," "for all x," and "for each x " as having the same meaning and symbolize each by "$(\forall x)$." Think of the symbol $\forall$ as an inverted A(representing all). If $P(x)$ is an open proposition with universe U, then $(\forall x)P(x)$ is a quantified proposition and is read as "every $x \in U$ has the property P."
b. The phrase "there exists an x " is called an **existential quantifier**. We regard "there exists an x," "for some x," and "for at least one x " as having the same meaning, and symbolize each by "$(\exists x)$." Think of the symbol $\exists$ as the backwards capital E(representing exists). If $P(x)$ is an open proposition with universe U, then $(\exists x)P(x)$ is a quantified proposition and is read as "there exists $x \in U$ with the property P."

Remarks:

i. To show that $(\forall x)P(x)$ is F, it is sufficient to find at least one $a \in U$ such that $P(a)$ is F. Such an element $a \in U$ is called a ***counter example***.
ii. $(\exists x)P(x)$ is F if we cannot find any $a \in U$ having the property P.

Example 1.14:

a. Write the following statements using quantifiers.
   i. For each real number $x > 0, x^2 + x - 6 = 0$.

Solution: $(\forall x > 0)(x^2 + x - 6 = 0)$.

ii. There is a real number $x > 0$ such that $x^2 + x - 6 = 0$.

Solution: $(\exists x > 0)(x^2 + x - 6 = 0)$.

iii. The square of any real number is nonnegative.

Solution: $(\forall x \in \mathbb{R})(x^2 \geq 0)$.

b.

i. Let $P(x): x^2 + 1 \geq 0$. The truth value for $(\forall x)P(x)$ [i.e $(\forall x)(x^2 + 1 \geq 0)$] is T.

ii. Let $P(x): x < x^2$. The truth value for $(\forall x)(x < x^2)$ is F. $x = \frac{1}{2}$ is a counterexample since $\frac{1}{2} \in \mathbb{R}$ but $\frac{1}{2} < \frac{1}{4}$. On the other hand, $(\exists x)P(x)$ is true, since $-1 \in \mathbb{R}$ such that $-1 < 1$.

iii. Let $P(x): |x| = -1$. The truth value for $(\exists x)P(x)$ is F since there is no real number whose absolute value is -1.

Relationship between the existential and universal quantifiers

If $P(x)$ is a formula in x, consider the following four statements.

a. $(\forall x)P(x)$.
b. $(\exists x)P(x)$.
c. $(\forall x)\neg P(x)$.
d. $(\exists x)\neg P(x)$.

We might translate these into words as follows.

a. Everything has property P.
b. Something has property P.
c. Nothing has property P.
d. Something does not have property P.

Now (d) is the denial of (a), and (c) is the denial of (b), on the basis of everyday meaning. Thus, for example, the existential quantifier may be defined in terms of the universal quantifier.

Now we proceed to discuss the negation of quantifiers. Let $P(x)$ be an open proposition. Then $(\forall x)P(x)$ is false only if we can find an individual "a" in the universe such that $P(a)$ is false. If we succeed in getting such an individual, then $(\exists x)\neg P(x)$ is true. Hence $(\forall x)P(x)$ will be false if $(\exists x)\neg P(x)$ is true. Therefore the negation of $(\forall x)P(x)$ is $(\exists x)\neg P(x)$. Hence we conclude that

$$\neg(\forall x)P(x) \equiv (\exists x)\neg P(x).$$

Similarly, we can easily verified that

$$\neg(\exists x)P(x) \equiv (\forall x)\neg P(x).$$

Remark: To negate a statement that involves the quantifiers $\forall$ and $\exists$, change each $\forall$ to $\exists$, change each $\exists$ to $\forall$, and negate the open statement.

Example 1.15:

Let $U = \mathbb{R}$.

a. $\neg(\exists x)(x < x^2) \equiv (\forall x)\neg(x < x^2)$
$\equiv (\forall x)(x \geq x^2)$.

b. $\neg(\forall x)(4x + 1 = 0) \equiv (\exists x)\neg(4x + 1 = 0)$
$\equiv (\exists x)(4x + 1 \neq 0)$.

Given propositions containing quantifiers we can form a compound proposition by joining them with connectives in the same way we form a compound proposition without quantifiers. For example, if we have $(\forall x)P(x)$ and $(\exists x)Q(x)$ we can form $(\forall x)P(x) \Leftrightarrow (\exists x)Q(x)$.

Consider the following statements involving quantifiers. Illustrations of these along with translations appear below.

a. All rationals are reals. $(\forall x)(\mathbb{Q}(x) \Longrightarrow \mathbb{R}(x))$.
b. No rationals are reals. $(\forall x)(\mathbb{Q}(x) \Longrightarrow \neg\mathbb{R}(x))$.
c. Some rationals are reals. $(\exists x)(\mathbb{Q}(x) \wedge \mathbb{R}(x))$.
d. Some rationals are not reals. $(\exists x)(\mathbb{Q}(x) \wedge \neg\mathbb{R}(x))$.

Example 1.16:

Let $U =$ The set of integers.

Let $P(x)$: x is a prime number.
$Q(x)$: x is an even number.
$R(x)$: x is an odd number.

Then

a. $(\exists x)[P(x) \Longrightarrow Q(x)]$ is T; since there is an x, say 2, such that $P(2) \Longrightarrow Q(2)$ is T.
b. $(\forall x)[P(x) \Longrightarrow Q(x)]$ is F. As a counterexample take 7. Then $P(7)$ is T and $Q(7)$ is F. Hence $P(7) \Longrightarrow Q(7)$.
c. $(\forall x)[R(x) \wedge P(x)]$ is F.
d. $(\forall x)[(R(x) \wedge P(x)) \Longrightarrow Q(x)]$ is F.

Quantifiers Occurring in Combinations

So far, we have only considered cases in which universal and existential quantifiers appear simply. However, if we consider cases in which universal and existential quantifiers occur in combination, we are lead to essentially new logical structures. The following are the simplest forms of combinations:

1. $(\forall x)(\forall y)P(x, y)$
   "for all x and for all y the relation $P(x, y)$ holds";
2. $(\exists x)(\exists y)P(x, y)$
   "there is an x and there is a y for which $P(x, y)$ holds";
3. $(\forall x)(\exists y)P(x, y)$
   "for every x there is a y such that $P(x, y)$ holds";

4. $(\exists x)(\forall y)P(x,y)$
   "there is an x which stands to every y in the relation $P(x,y)$."

Example 1.17:

Let U = The set of integers.

Let $P(x,y): x + y = 5$.

a. $(\exists x)\ (\exists y)\ P(x,y)$ means that there is an integer x and there is an integer y such that $x + y = 5$. This statement is true when $x = 4$ and $y = 1$, since $4 + 1 = 5$. Therefore, the statement $(\exists x)\ (\exists y)\ P(x,y)$ is always true for this universe. There are other choices of x and y for which it would be true, but the symbolic statement merely says that there is at least one choice for x and y which will make the statement true, and we have demonstrated one such choice.

b. $(\exists x)\ (\forall y)\ P(x,y)$ means that there is an integer x_0 such that for every y, $x_0 + y = 5$. This is false since no fixed value of x_0 will make this true for all y in the universe; e.g. if $x_0 = 1$, then $1 + y = 5$ is false for some y.

c. $(\forall x)\ (\exists y)\ P(x,y)$ means that for every integer x, there is an integer y such that $x + y = 5$. Let $x = a$, then $y = 5 - a$ will always be an integer, so this is a true statement.

d. $(\forall x)\ (\forall y)\ P(x,y)$ means that for every integer x and for every integer y, $x + y = 5$. This is false, for if $x = 2$ and $y = 7$, we get $2 + 7 = 9 \neq 5$.

Example 1.18:

a. Consider the statement

$$\text{For every two real numbers } x \text{ and } y, x^2 + y^2 \geq 0.$$

   If we let

$$P(x,y): x^2 + y^2 \geq 0$$

where the domain of both x and y is $\mathbb{R}$, the statement can be expressed as

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})P(x,y) \text{ or as } (\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x^2 + y^2 \geq 0).$$

Since $x^2 \geq 0$ and $y^2 \geq 0$ for all real numbers x and y, it follows that $x^2 + y^2 \geq 0$ and so $P(x,y)$ is true for all real numbers x and y. Thus the quantified statement is true.

b. Consider the open statement

$$P(x,y): |x - 1| + |y - 2| \leq 2$$

where the domain of the variable x is the set E of even integers and the domain of the variable y is the set O of odd integers. Then the quantified statement

$$(\exists x \in E)(\exists y \in O)P(x,y)$$

can be expressed in words as

There exist an even integer x and an odd integer y such that $|x-1|+|y-2| \leq 2$.

Since $P(2,3)\colon 1+1 \leq 2$ is true, the quantified statement is true.

c. Consider the open statement

$$P(x,y)\colon xy = 1$$

where the domain of both x and y is the set $\mathbb{Q}^+$ of positive rational numbers. Then the quantified statement

$$(\forall x \in \mathbb{Q}^+)(\exists y \in \mathbb{Q}^+)P(x,y)$$

can be expressed in words as

For every positive rational number x, there exists a positive rational number y such that $xy = 1$. It turns out that the quantified statement is true. If we replace $\mathbb{Q}^+$ by $\mathbb{R}$, then we have

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})P(x,y) .$$

Since $x = 0$ and for every real number $y, xy = 0 \neq 1$, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})P(x,y)$ is false.

d. Consider the open statement

$$P(x,y)\colon xy \text{ is odd}$$

where the domain of both x and y is the set $\mathbb{N}$ of natural numbers. Then the quantified statement

$$(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})P(x,y),$$

expressed in words, is

There exists a natural number x such that for every natural numbers y, xy is odd. The statement is false.

In general, from the meaning of the universal quantifier it follows that in an expression $(\forall x)(\forall y)P(x,y)$ the two universal quantifiers may be interchanged without altering the sense of the sentence. This also holds for the existential quantifies in an expression such as $(\exists x)(\exists y)P(x,y)$.

In the statement $(\forall x)(\exists y)P(x,y)$, the choice of y is allowed to depend on x - the y that works for one x need not work for another x. On the other hand, in the statement $(\exists y)(\forall x)P(x,y)$, the y must work for all x, i.e., y is independent of x. For example, the expression $(\forall x)(\exists y)(x < y)$, where x and y are variables referring to the domain of real numbers, constitutes a true proposition, namely, "For every number x, there is a number y, such that x is less that y," i.e., "given any number, there is a greater number." However, if the order of the symbol $(\forall x)$ and $(\exists y)$ is changed, in this case, we obtain: $(\exists y)(\forall x)(x < y)$, which is a false proposition, namely, "There is a number which is greater than every number." By transposing $(\forall x)$ and $(\exists y)$, therefore, we get a different statement.

The logical situation here is:

$$(\exists y)(\forall x)P(x,y) \Rightarrow (\forall x)(\exists y)P(x,y).$$

Finally, we conclude this section with the remark that there are no mechanical rules for translating sentences from English into the logical notation which has been introduced. In every case one must first decide on the meaning of the English sentence and then attempt to convey that same meaning in terms of predicates, quantifiers, and, possibly, individual constants.

Exercises

1. In each of the following, two open statements $P(x,y)$ and $Q(x,y)$ are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x,y) \Longrightarrow Q(x,y)$ for the given values of x and y.
   a. $P(x,y): x^2 - y^2 = 0$. and $Q(x,y): x = y$. $(x,y) \in \{(1,-1),(3,4),(5,5)\}$.
   b. $P(x,y): |x| = |y|$. and $Q(x,y): x = y$. $(x,y) \in \{(1,2),(2,-2),(6,6)\}$.
   c. $P(x,y): x^2 + y^2 = 1$. and $Q(x,y): x + y = 1$.
      $(x,y) \in \{(1,-1),(-3,4),(0,-1),(1,0)\}$.
2. Let O denote the set of odd integers and let $P(x): x^2 + 1$ is even, and $Q(x): x^2$ is even. be open statements over the domain O. State $(\forall x \in O)P(x)$ and $(\exists y \in O)Q(x)$ in words.
3. State the negation of the following quantified statements.
   a. For every rational number r, the number $\frac{1}{r}$ is rational.
   b. There exists a rational number r such that $r^2 = 2$.
4. Let $P(n): \frac{5n-6}{3}$ is an integer. be an open sentence over the domain $\mathbb{Z}$. Determine, with explanations, whether the following statements are true or false:
   a. $(\forall n \in \mathbb{Z})P(n)$.
   b. $(\exists n \in \mathbb{Z})P(n)$.
5. Determine the truth value of the following statements.
   a. $(\exists x \in \mathbb{R})(x^2 - x = 0)$.
   b. $(\forall x \in \mathbb{N})(x + 1 \geq 2)$.
   c. $(\forall x \in \mathbb{R})(\sqrt{x^2} = x)$.
   d. $(\exists x \in \mathbb{Q})(3x^2 - 27 = 0)$.
   e. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y + 3 = 8)$.
   f. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 + y^2 = 9)$.
   g. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 5)$.
   h. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 5)$
6. Consider the quantified statement

   For every $x \in A$ and $y \in A$, $xy - 2$ is prime.

   where the domain of the variables x and y is $A = \{3,5,11\}$.
   a. Express this quantified statement in symbols.
   b. Is the quantified statement in (a) true or false? Explain.
   c. Express the negation of the quantified statement in (a) in symbols.
   d. Is the negation of the quantified in (a) true or false? Explain.
7. Consider the open statement $P(x,y): \frac{x}{y} < 1$. where the domain of x is $A = \{2,3,5\}$ and the

domain of y is $B = \{2,4,6\}$.

a. State the quantified statement $(\forall x \in A)(\exists y \in B)P(x,y)$ in words.

b. Show quantified statement in (a) is true.

8. Consider the open statement $P(x,y): x - y < 0$. where the domain of x is $A = \{3,5,8\}$ and the domain of y is $B = \{3,6,10\}$.

a. State the quantified statement $(\exists y \in B)(\forall x \in A)P(x,y)$ in words.

b. Show quantified statement in (a) is true.

1. 3. Argument and Validity

Section objectives:

After completing this section, students will be able to:-

- ✓ Define argument (or logical deduction).
- ✓ Identify hypothesis and conclusion of a given argument.
- ✓ Determine the validity of an argument using a truth table.
- ✓ Determine the validity of an argument using rules of inferences.

Definition 1.7: An argument (logical deduction) is an assertion that a given set of statements $p_1, p_2, p_3, \dots, p_n$, called **hypotheses** or ***premises***, yield another statement Q, called the ***conclusion***. Such a logical deduction is denoted by:

$$p_1, p_2, p_3, \dots, p_n \vdash Q \text{ or}$$

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \underline{p_n} \\ Q \end{array}$$

Example 1.19: Consider the following argument:

If you study hard, then you will pass the exam.
You did not pass the exam.
Therefore, you did not study hard.
Let p: You study hard.
q: You will pass the exam.
The argument form can be written as:

$\underline{\neg q}$

When is an argument form accepted to be correct? In normal usage, we use an argument in order to demonstrate that a certain conclusion follows from known premises. Therefore, we shall require that under any assignment of truth values to the statements appearing, if the premises became all true, then the conclusion must also become true. Hence, we state the following definition.

Definition 1.8: An argument form $p_1, p_2, p_3, \dots, p_n \vdash Q$ is said to be ***valid*** if Q is true whenever all the premises $p_1, p_2, p_3, \dots, p_n$ are true; otherwise it is ***invalid***.

Example 1.20: Investigate the validity of the following argument:

a. $p \Rightarrow q, \neg q \mid\!- \neg p$

b. $p \Rightarrow q, \neg q \Rightarrow r \mid\!- p$

c. If it rains, crops will be good. It did not rain. Therefore, crops were not good.

Solution: First we construct a truth table for the statements appearing in the argument forms.

a.

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$
T	*T*	*F*	*F*	*T*
T	*F*	*F*	*T*	*F*
F	*T*	*T*	*F*	*T*
F	*F*	*T*	*T*	*T*

The premises $p \Rightarrow q$ and $\neg q$ are true simultaneously in row 4 only. Since in this case $\neg p$ is also true, the argument is valid.

b.

p	q	r	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow r$
T	*T*	*T*	*F*	*T*	*T*
T	*T*	*F*	*F*	*T*	*T*
T	*F*	*T*	*T*	*F*	*T*
T	*F*	*F*	*T*	*F*	*F*
F	*T*	*T*	*F*	*T*	*T*
F	*T*	*F*	*F*	*T*	*T*
F	*F*	*T*	*T*	*T*	*T*
F	*F*	*F*	*T*	*T*	*F*

The 1st, 2nd, 5th, 6th and 7th rows are those in which all the premises take value T. In the 5th, 6th and 7th rows however the conclusion takes value F. Hence, the argument form is invalid.

c. Let p: It rains.

q: Crops are good.

$\neg p$: It did not rain.

$\neg q$: Crops were not good.

The argument form is $p \Longrightarrow q, \neg p \vdash \neg q$

Now we can use truth table to test validity as follows:

p	q	$\neg p$	$\neg q$	$p \Longrightarrow q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

The premises $p \Longrightarrow q$ and $\neg p$ are true simultaneously in row 4 only. Since in this case $\neg q$ is also true, the argument is valid.

Remark:

1. What is important in validity is the form of the argument rather than the meaning or content of the statements involved.
2. The argument form $p_1, p_2, p_3, \ldots, p_n \vdash Q$ is valid iff the statement $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \Longrightarrow Q$ is a tautology.

Rules of inferences

Below we list certain valid deductions called rules of inferences.

1. Modes Ponens

$$\begin{array}{l} p \\ \underline{p \Longrightarrow q} \\ q \end{array}$$

2. Modes Tollens

$$\begin{array}{l} \neg q \\ \underline{p \Longrightarrow q} \\ \neg p \end{array}$$

3. Principle of Syllogism

$$\begin{array}{l} p \Longrightarrow q \\ \underline{q \Longrightarrow r} \\ p \Longrightarrow r \end{array}$$

4. Principle of Adjunction

a.
$$\frac{\begin{array}{c} p \\ q \end{array}}{p \wedge q}$$

b.
$$\frac{q}{p \vee q}$$

5. Principle of Detachment

$$\frac{p \wedge q}{p, q}$$

6. Modes Tollendo Ponens

$$\frac{\begin{array}{c} \neg p \\ p \vee q \end{array}}{q}$$

7. Modes Ponendo Tollens

$$\frac{\begin{array}{c} \neg(p \wedge q) \\ p \end{array}}{\neg q}$$

8. Constructive Dilemma

$$\frac{\begin{array}{c} (p \Rightarrow q) \wedge (r \Rightarrow s) \\ p \vee r \end{array}}{q \vee s}$$

9. Principle of Equivalence

$$\frac{\begin{array}{c} p \Leftrightarrow q \\ p \end{array}}{q}$$

10. Principle of Conditionalization

$$\frac{p}{q \Rightarrow p}$$

Formal proof of validity of an argument

Definition 1.9: A formal proof of a conclusion Q given hypotheses $p_1, p_2, p_3, \dots, p_n$ is a sequence of stapes, each of which applies some inference rule to hypotheses or previously proven statements (antecedent) to yield a new true statement (the consequent).

A formal proof of validity is given by writing on the premises and the statements which follows from them in a single column, and setting off in another column, to the right of each statement, its justification. It is convenient to list all the premises first.

Example 1.21: Show that $p \Rightarrow \neg q, q \vdash \neg p$ is valid.

Solution:

1. q is true — premise
2. $p \Rightarrow \neg q$ is true — premise
3. $q \Rightarrow \neg p$ is true — contrapositive of (2)
4. ˥p is true — Modes Ponens using (1) and (3)

Example 1.22: Show that the hypotheses

It is not sunny this afternoon and it is colder than yesterday.
If we go swimming, then it is sunny.
If we do not go swimming, then we will take a canoe trip.
If we take a canoe trip, then we will be home by sunset.

Lead to the conclusion:

We will be home by sunset.

Let p: It is sunny this afternoon.
q: It is colder than yesterday.
r: We go swimming.
s: We take a canoe trip.
t: We will be home by sunset.

Then

1. $\neg p \land q$ — hypothesis
2. $\neg p$ — simplification using (1)
3. $r \Rightarrow p$ — hypothesis
4. $\neg r$ — Modus Tollens using (2) and (3)
5. $\neg r \Rightarrow s$ — hypothesis
6. s — Modus Ponens using (4) and (5)
7. $s \Rightarrow t$ — hypothesis
8. t — Modus Ponens using (6) and (7)

Exercises

1. Use the truth table method to show that the following argument forms are valid.
   i. $\neg p \Rightarrow \neg q, q \vdash p$.
   ii. $p \Rightarrow \neg p, p, r \Rightarrow q \vdash \neg r$.
   iii. $p \Rightarrow q, \neg r \Rightarrow \neg q \vdash \neg r \Rightarrow \neg p$.
   iv. $\neg r \lor \neg s, (\neg s \Rightarrow p) \Rightarrow r \vdash \neg p$.
   v. $p \Rightarrow q, \neg p \Rightarrow r, r \Rightarrow s \vdash \neg q \Rightarrow s$.
2. For the following argument given a, b and c below:
   i. Identify the premises.
   ii. Write argument forms.

iii. Check the validity.

a. If he studies medicine, he will get a good job. If he gets a good job, he will get a good wage. He did not get a good wage. Therefore, he did not study medicine.

b. If the team is late, then it cannot play the game. If the referee is here, then the team is can play the game. The team is late. Therefore, the referee is not here.

c. If the professor offers chocolate for an answer, you answer the professor's question. The professor offers chocolate for an answer. Therefore, you answer the professor's question

3. Give formal proof to show that the following argument forms are valid.

a. $\neg p \Rightarrow \neg q, q \vdash p$.

b. $p \Rightarrow \neg q, p, r \Rightarrow q \vdash \neg r$.

c. $p \Rightarrow q, \neg r \Rightarrow \neg q \vdash \neg r \Rightarrow \neg p$.

d. $\neg r \wedge \neg s, (\neg s \Rightarrow p) \Rightarrow r \vdash \neg p$.

e. $p \Rightarrow, \neg p \Rightarrow r, r \Rightarrow s \vdash \neg q \Rightarrow s$.

f. $\neg p \vee q, r \Rightarrow p, r \vdash q$.

g. $\neg p \wedge \neg q, (q \vee r) \Rightarrow p \vdash \neg r$.

h. $p \Rightarrow (q \vee r), \neg r, p \vdash q$.

i. $\neg q \Rightarrow \neg p, r \Rightarrow p, \neg q \vdash r$.

4. Prove the following are valid arguments by giving formal proof.

a. If the rain does not come, the crops are ruined and the people will starve. The crops are not ruined or the people will not starve. Therefore, the rain comes.

b. If the team is late, then it cannot play the game. If the referee is here then the team can play the game. The team is late. Therefore, the referee is not here.

1.4. Set theory

In this section, we study some part of set theory especially description of sets, Venn diagrams and operations of sets.

Section objectives:

After completing this section, students will be able to:-

- ✓ Explain the concept of set.
- ✓ Describe sets in different ways.
- ✓ Identify operations on sets.
- ✓ Illustrate sets using Venn diagrams.

1.4.1. The concept of a set

The term set is an undefined term, just as a point and a line are undefined terms in geometry. However, the concept of a set permeates every aspect of mathematics. Set theory underlies the language and concepts of modern mathematics. The term set refers to a well-defined collection of objects that share a certain property or certain properties. The term "**well-defined**" here means that the set is described in such a way that one can decide whether or not a given object belongs in the set. If A is a set, then the objects of the collection A are called the elements or members of the set A. If x is an element of the set A, we write $x \in A$. If x is not an element of the set A, we write $x \notin A$.

As a convention, we use capital letters to denote the names of sets and lowercase letters for elements of a set.

Note that for each objects x and each set A, exactly one of $x \in A$ or $x \notin A$ but not both must be true.

1.4.2. Description of sets

Sets are described or characterized by one of the following four different ways.

1. **Verbal Method**

In this method, an ordinary English statement with minimum mathematical symbolization of the property of the elements is used to describe a set. Actually, the statement could be in any language.

Example 1.23:

a. The set of counting numbers less than ten.

b. The set of letters in the word "Addis Ababa."

c. The set of all countries in Africa.

2. **Roster/Complete Listing Method**

If the elements of a set can all be listed, we list them all between a pair of braces without repetition separating by commas, and without concern about the order of their appearance. Such a method of describing a set is called ***the roster/complete listing*** method.

Examples 1.24:

a. The set of vowels in English alphabet may also be described as $\{a, e, i, o, u\}$.

b. The set of positive factors of 24 is also described as $\{1, 2, 3, 4, 6, 8, 12, 24\}$.

Remark:

i. We agree on the convention that the order of writing the elements in the list is immaterial. As a result the sets $\{a, b, c\}, \{b, c, a\}$ and $\{c, a, b\}$ contain the same elements, namely a, b and c.

ii. The set $\{a, a, b, b, b\}$ contains just two distinct elements; namely a and b, hence it is the same set as $\{a, b\}$. We list distinct elements without repetition.

Example 1.25:

a. Let $A = \{a, b, \{c\}\}$. Elements of A are a, b and $\{c\}$.
Notice that c and $\{c\}$ are different objects. Here $\{c\} \in A$ but $c \notin A$.

b. Let $B = \{\{a\}\}$. The only element of B is $\{a\}$. But $a \notin B$.

c. Let $C = \{a, b, \{a, b\}, \{a, \{a\}\}\}$. Then C has four elements.

The readers are invited to write down all the elements of C.

3. **Partial Listing Method**

In many occasions, the number of elements of a set may be too large to list them all; and in other occasions there may not be an end to the list. In such cases we look for a common property of the elements and describe the set by partially listing the elements. More precisely, if the common property is simple that it can easily be identified from a list of the first few elements, then with in a pair of braces, we list these few elements followed (or preceded) by exactly three dotes and possibly by one last element. The following are such instances of describing sets by partial listing method.

Example 1.26:

a. The set of all counting numbers is $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.
b. The set of non-positive integers is $\{\ldots, -4, -3, -2, -1, 0\}$.
c. The set of multiples of 5 is $\{\ldots, -15, -10, -5, 0\ 5, 10, 15, \ldots\}$.
d. The set of odd integers less than 100 is $\{\ldots, -3, -1, 1, 3, 5, \ldots\ 99\}$.

4. **Set-builder Method**

When all the elements satisfy a common property P, we express the situation as an open proposition $P(x)$ and describe the set using a method called the ***Set-builder Method*** as follows:

$$A = \{x \mid P(x)\}\ or\ A = \{x : P(x)\}$$

We read it as "A is equal to the set of all x's such that $P(x)$ is true." Here the bar "| " and the colon ":" mean "such that." Notice that the letter x is only a place holder and can be replaced throughout by other letters. So, for a property P, the set $\{x \mid P(x)\}, \{t \mid P(t)\}$ and $\{y \mid P(y)\}$ are all the same set.

Example 1.27: The following sets are described using the set-builder method.

a. $A = \{x \mid x \text{ is a vowel in the English alphabet}\}$.
b. $B = \{t \mid t \text{ is an even integer}\}$.
c. $C = \{n \mid n \text{ is a natural number and } 2n - 15 \text{ is negative}\}$.
d. $D = \{y \mid y^2 - y - 6 = 0\}$.
e. $E = \{x \mid x \text{ is an integer and } x - 1 < 0 \Rightarrow x^2 - 4 > 0\}$.

Exercise: Express each of the above by using either the complete or the partial listing method.

Definition 1.10: The set which has no element is called the empty (or null) set and is denoted by ϕ or $\{\}$.

Example 1.28: The set of $x \in \mathbb{R}$ such that $x^2 + 1 = 0$ is an empty set.

Definition 1.11: A set is finite if it has limited number of elements and it is called infinite if it has unlimited number of elements.

Relationships between two sets

Definition 1.12: Set B is said to be a ***subset*** of set A (or is contained in A), denoted by $B \subseteq A$, if every element of B is an element of A, i.e.,

$$(\forall x)(x \in B \Longrightarrow x \in A).$$

It follows from the definition that set B is not a subset of set A if at least one element of B is not an element of A. i.e., $B \nsubseteq A \Leftrightarrow (\exists x)(x \in B \Longrightarrow x \notin A)$. In such cases we write $B \nsubseteq A$ or $A \nsupseteq B$.

Remarks: For any set $A, \phi \subseteq A$ and $A \subseteq A$.

Example 1.29:

a. If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{a, b, d\}$, then $A \subseteq B$ and $A \subseteq C$. On the other hand, it is clear that: $B \nsubseteq A$, $B \nsubseteq C$ and $C \nsubseteq B$.

b. If $S = \{x \mid x \text{ is a multiple of } 6\}$ and $T = \{x \mid x \text{ is even integer}\}$, then $S \subseteq T$ since every multiple of 6 is even. However, $2 \in T$ while $2 \notin S$. Thus $T \nsubseteq S$.

c. If $A = \{a, \{b\}\}$, then $\{a\} \subseteq A$and $\{\{b\}\} \subseteq A$. On the other hand, since $b \notin A$, $\{b\} \nsubseteq A$, and $\{a, b\} \nsubseteq A$.

Definition 1.13:

a. Sets A and B are said to be ***equal*** if they contain exactly the same elements. In this case, we write $A = B$. That is, $(\forall x)(x \in B \Leftrightarrow x \in A)$.

b. Sets A and B are said to be ***equivalent*** if and only if there is a one to one correspondence among their elements. In this case, we write $A \leftrightarrow B$.

Example 1.30:

a. The sets $\{1, 2, 3\}, \{2, 1, 3\}, \{1, 3, 2\}$ are all equal.

b. $\{x | xisacountingnumber\} = \{x | xisapositiveinteger\}$

Definition 1.14: Set A is said to be a **proper subset** of set B if every element of A is also an element of B, but B has at least one element that is not in A. In this case, we write $A \subset B$. We also say B is a proper super set of A, and write $B \supset A$. It is clear that

$$A \subset B \Leftrightarrow [(\forall x)(x \in A \Longrightarrow x \in B) \wedge (A \neq B)].$$

Remark: Some authors do not use the symbol $\subseteq$. Instead they use the symbol $\subset$ for both subset and proper subset. In this material, we prefer to use the notations commonly used in high school mathematics, and we continue using $\subseteq$ and $\subset$ differently, namely for subset and proper subset, respectively.

Definition 1.15: Let A be a set. The power set of A, dented by $P(A)$, is the set whose elements are all subsets of A. That is,

$$P(A) = \{B: B \subseteq A\}.$$

Note: If a set A is finite with n elements, then

a. The number of subsets of A is 2^n and
b. The number proper subsets of A is $2^n - 1$.

Example 1.31: Let $A = \{x, y, z\}$. As noted before, ϕ and A are subset of A. Moreover, $\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}$ and $\{y, z\}$ are also subsets of A. Therefore,

$$P(A) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, A\}.$$

Frequently it is necessary to limit the topic of discussion to elements of a certain fixed set and regard all sets under consideration as a subset of this fixed set. We call this set the **universal set** or the **universe** and denoted by $\boldsymbol{U}$.

Exercises

1. Which of the following are sets?
   c. 1,2,3
   d. {1,2},3
   e. {{1},2},3
   f. {1,{2},3}
   g. {1,2,a,b}.
2. Which of the following sets can be described in complete listing, partial listing and/or set-builder methods? Describe each set by at least one of the three methods.
   a. The set of the first 10 letters in the English alphabet.
   b. The set of all countries in the world.
   c. The set of students of Addis Ababa University in the 2018/2019 academic year.
   d. The set of positive multiples of 5.
   e. The set of all horses with six legs.
3. Write each of the following sets by listing its elements within braces.
   c. $A = \{x \in \mathbb{Z}: -4 < x \leq 4\}$
   d. $B = \{x \in \mathbb{Z}: x^2 < 5\}$
   e. $C = \{x \in \mathbb{N}: x^3 < 5\}$
   f. $D = \{x \in \mathbb{R}: x^2 - x = 0\}$

g. $E = \{x \in \mathbb{R}: x^2 + 1 = 0\}$.

4. Let A be the set of positive even integers less than 15. Find the truth value of each of the following.
   a. $15 \in A$
   b. $-16 \in A$
   c. $\phi \in A$
   d. $12 \subset A$
   e. $\{2, 8, 14\} \in A$
   f. $\{2,3,4\} \subseteq A$
   g. $\{2,4\} \in A$
   h. $\phi \subset A$
   i. $\{246\} \subseteq A$

5. Find the truth value of each of the following and justify your conclusion.
   a. $\phi \subseteq \phi$
   b. $\{1,2\} \subseteq \{1,2\}$
   c. $\phi \in A$ for any set A
   d. $\{\phi\} \subseteq A$, for any set A
   e. $5, 7 \subseteq \{5, 6, 7, 8\}$
   f. $\phi \in \{\{\phi\}\}$
   g. For any set $A, A \subset A$
   h. $\{\phi\} = \phi$
6. For each of the following set, find its power set.
   a. $\{ab\}$
   b. $\{1, 1.5\}$
   c. $\{a, b\}$
   d. $\{a, 0.5, x\}$
7. How many subsets and proper subsets do the sets that contain exactly $1, 2, 3, 4, 8, 10$ and 20 elements have?
8. Is there a set A with exactly the following indicated property?
   a. Only one subset
   b. Only one proper subset
   c. Exactly 3 proper subsets
   d. Exactly 4 subsets
   e. Exactly 6 proper subsets
   f. Exactly 30 subsets
   g. Exactly 14 proper subsets
   h. Exactly 15 proper subsets

9. How many elements does A contain if it has:
    a. 64 subsets?
    b. 31 proper subsets?
    c. No proper subset?
    d. 255 proper subsets?
10. Find the truth value of each of the following.
    a. $\phi \in P(\phi)$
    b. $For\ any\ set\ A, \phi \subseteq P(A)$
    c. For any set $A, A \in P(A)$
    d. For any set $A, A \subset P(A)$.
11. For any three sets A, B and C, prove that:
    a. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
    b. If $A \subset B$ and $B \subset C$, then $A \subset C$.

1.4.3. Set Operations and Venn diagrams

Given two subsets A and B of a universal set U, new sets can be formed using A and B in many ways, such as taking common elements or non-common elements, and putting everything together. Such processes of forming new sets are called ***set operations***. In this section, three most important operations, namely union, intersection and complement are discussed.

Definition 1.16: The union of two sets A and B, denoted by $A \cup B$, is the set of all elements that are either in A or in B (or in both sets). That is,

$$A \cup B = \{x: (x \in A) \vee (x \in B)\}.$$

As easily seen the union operator "∪" in the theory of set is the counterpart of the logical operator "∨".

Definition 1.17: The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements that are in A and B. That is,

$$A \cap B = \{x: (x \in A) \wedge (x \in B)\}.$$

As suggested by definition 1.17, the intersection operator "∩" in the theory of sets is the counterpart of the logical operator "∧".

Note: - Two sets A and B are said to be disjoint sets if $A \cap B = \phi$.

Example 1.32:

a. Let $A = \{0, 1, 3, 5, 6\}$ and $B = \{1, 2, 3, 4, 6, 7\}$. Then, $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $A \cap B = \{1, 3, 6\}$.

b. Let A = The set of positive even integers, and
B = The set of positive multiples of 3. Then,

$A \cup B = \{x: x \text{ is a positive intger that is either even or a multiple of } 3\}$
$= \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, \dots\}$

$A \cap B = \{x \mid x \text{ is a positive integer that is both even and multiple of } 3\}$
$= \{6, 12, 18, 24, \dots\}$
$= \{x \mid x \text{ is a positive multiple of } 6.\}$

Definition 1.18: The difference between two sets A and B, denoted by $A - B$, is the of all elements in A and not in B; this set is also called the **relative complement** of B with respect to A. Symbolically,

$$A - B = \{x: x \in A \wedge x \notin B\}.$$

Note: $A - B$ is sometimes denoted by $A \backslash B$. $A - B$ and $A \backslash B$ are used interchangeably.

Example 1.33: If $= \{1,3,5\}$, $B = \{1,2\}$, then $A - B = \{3,5\}$ and $B - A = \{2\}$.
Note: The above example shows that, in general, $A - B$ are $B - A$ disjoint.

Definition 1.19: Let A be a subset of a universal set U. The ***absolute complement*** (or simply ***complement***) of A, denoted by A' (or A^cor $\overline{A}$), is defined to be the set of all elements of U that are not in A. That is,

$A' = \{x: x \in U \wedge x \notin A\}$ or $x \in A' \Leftrightarrow x \notin A \Leftrightarrow \neg(x \in A)$.

Notice that taking the absolute complement of A is the same as finding the relative complement of A with respect to the universal set U. That is,

$$A' = U - A.$$

Example 1.34:

a. If $U = \{0,1,2,3,4\}$, and if $A = \{3,4\}$, then $A' = \{0,1,2\}$.
b. Let $U = \{1, 2, 3, \dots, 12\}$
$A = \{x \mid x \text{ is a positive factor of } 12\}$
and $B = \{x \mid x \text{ is an odd integer in } U\}$.
Then, $A' = \{5, 7, 8, 9, 10, 11\}$, $B' = \{2, 4, 6, 8, 10, 12\}$,
$(A \cup B)' = (8, 10\}$, $A' \cup B' = \{2, 4, 5, 6, \dots, 12\}$,
$A' \cap B' = \{8, 10\}$, and $(A \backslash B)' = \{1, 3, 5, 7, 8, 9, 10, 11\}$.
c. Let $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, e, g, h\}$ and
$B = \{b, c, e, f, h\}$. Then
$A' = \{b, c, d, f\}$, $B' = \{a, d, g\}$, $B - A = \{b, c, f\}$,
$A - B = \{a, g\}$, and $(A \cup B)' = \{d\}$.

Find $(A \cap B)'$, $A' \cap B'$, $A' \cup B'$. Which of these are equal?

Theorem 1.1: For any two sets A and B, each of the following holds.

1. $(A')' = A$.
2. $A' = U - A$.
3. A– B = A ∩ B′A– B = A∩B'$A - B = A \cap B'$.
4. $(A \cup B)' = A' \cap B'$.
5. $(A \cap B)' = A' \cup B'$.
6. $A \subseteq B \Leftrightarrow B' \subseteq A'$.

Now we define the symmetric difference of two sets.

Definition 1.20: The symmetric difference of two sets A and B, denoted by $A\Delta B$, is the set

$$A\Delta B = (A - B) \cup (B - A).$$

Example 1.35: Let $U = \{1,2,3, \ldots ,10\}$ be the universal set, $A = \{2,4,6,8,9,10\}$ and $B = \{3,5,7,9\}$. Then $B - A = \{3,5,7\}$ and $A - B = \{2,4,6,8,10\}$. Thus $A \Delta B = \{2,3,4,5,6,7,8,10\}$.

Theorem 1.2: For any three sets A, B and C, each of the following holds.

a. $A \cup B = B \cup A$. (∪ is commutative)
b. $A \cap B = B \cap A$. (∩ is commutative)
c. $(A \cup B) \cup C = A \cup (B \cup C)$. (∪ is associative)
d. $(A \cap B) \cap C = A \cap (B \cap C)$. (∩ is associative)
e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (∪ is distributive over ∩)
f. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (∩ is distributive over ∪)

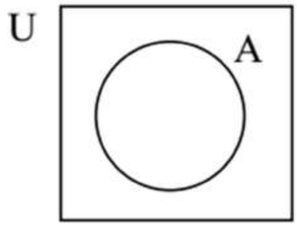
Let us prove property "e" formally.

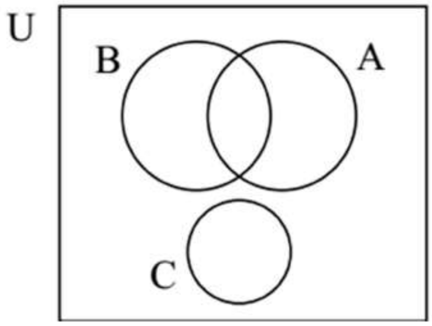
$x \in A \cup (B \cap C) \Leftrightarrow (x \in A) \vee (x \in B \cap C)$ (definition of ∪)
$\Leftrightarrow x \in A \vee (x \in B \wedge x \in C)$ (definition of ∩)
$\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$ (∨ is distributive over ∧)
$\Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup C)$ (definition of ∪)
$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$ (definition of ∩)

Therefore, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
The readers are invited to prove the rest part of theorem (1.2).

Venn diagrams

While working with sets, it is helpful to use diagrams, called **Venn diagrams,** to illustrate the relationships involved. A Venn diagram is a schematic or pictorial representative of the sets involved in the discussion. Usually sets are represented as interlocking circles, each of which is enclosed in a rectangle, which represents the universal set U.

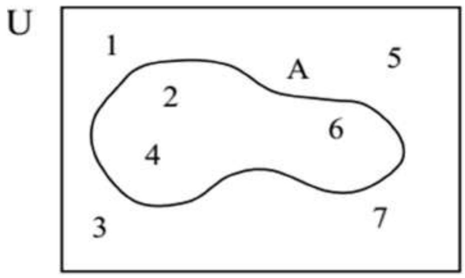




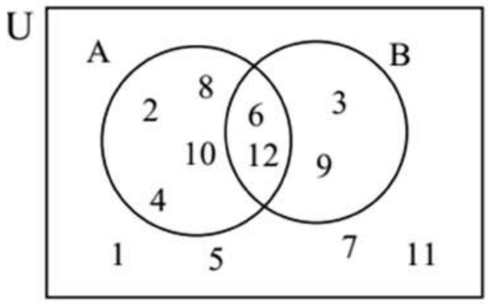
In some occasions, we list the elements of set A inside the closed curve representing A.

Example 1.36:

a. If $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{2, 4, 6\}$, then a Venn diagram representation of these two sets looks like the following.

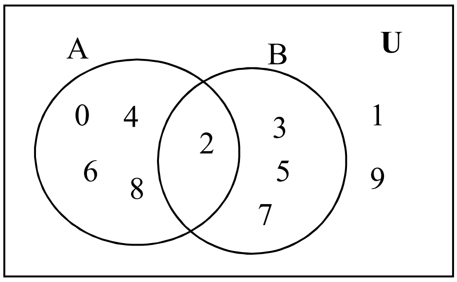


b. Let $U = \{x \mid x \text{ is a positive integer less than } 13\}$
$A = \{x \mid x \in U \text{ and } x \text{ is even}\}$
$B = \{x \mid x \in U \text{ and } x \text{ is a multiple of } 3\}$.
A Venn diagram representation of these sets is given below.

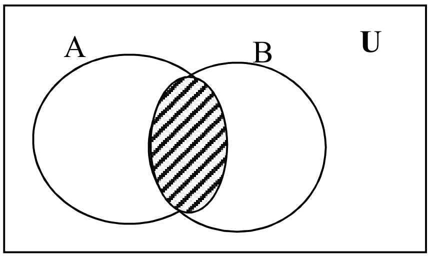


Example 1.37: Let U = The set of one digits numbers
A = The set of one digits even numbers
B = The set of positive prime numbers less than 10

We illustrate the sets using a Venn diagram as follows.

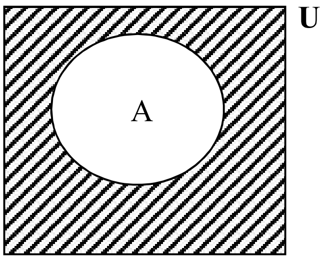


a. Illustrate $A \cap B$ by a Venn diagram



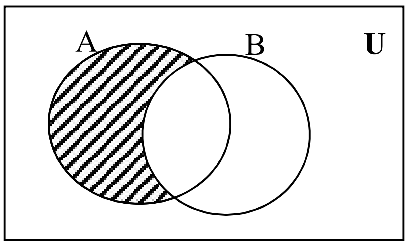
$A \cap B$: The shaded portion

b. Illustrate A' by a Venn diagram



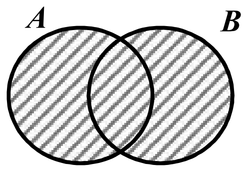
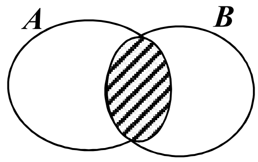
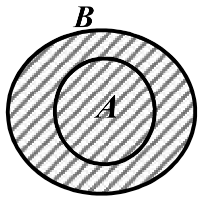
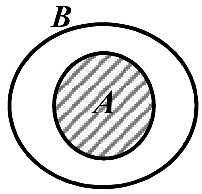
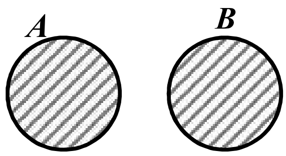
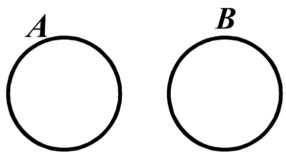
A' : The shaded portion

c. Illustrate $A \backslash B$ by using a Venn diagram



$A \setminus B$: The shaded portion

Now we illustrate intersections and unions of sets by Venn diagram.

Cases	Shaded is $A \cup B$	Shaded $A \cap B$
Only some common elements		
$A \subseteq B$		
No common element		 $A \cap B = \varnothing$

Exercises

1. If $B \subseteq A$, $A \cap B' = \{1,4,5\}$ and $A \cup B = \{1,2,3,4,5,6\}$, find B.
2. Let $A = \{2,4,6,7,8,9\}$,
   $B = \{1,3,5,6,10\}$ and
   $C = \{ x: 3x + 6 = 0$ or $2x + 6 = 0\}$. Find
   a. $A \cup B$.
   b. Is $(A \cup B) \cup C = A \cup (B \cup C)$?
3. Suppose $U =$ The set of one digit numbers and
   $A = \{x: x$ is an even natural number less than or equal to $9\}$
   Describe each of the sets by complete listing method:
   a. A'.
   b. $A \cap A'$.
   c. $A \cup A'$.
   d. $(A')'$
   e. $\phi - U$.
   f. ϕ'
   g. U'
4. Suppose $U =$ The set of one digit numbers and

$A = \{x: x$ is an even natural number less than or equal to $9\}$

Describe each of the sets by complete listing method:

h. A'.

i. $A \cap A'$.

j. $A \cup A'$.

k. $(A')'$

l. $\phi - U$.

m. ϕ'

n. U'

5. Use Venn diagram to illustrate the following statements:

   a. $(A \cup B)' = A' \cap B'$.

   b. $(A \cap B)' = A' \cup B'$.

   c. If $A \nsubseteq B$, then $A \backslash B \neq \phi$.

   d. $A \cup A' = U$.

6. Let $A = \{1,2,3,4\}, B = \{5,7,8,9\}$ and $C = \{6,7,8\}$. Then show that $(A\backslash B)\backslash C = A\backslash(B\backslash C)$.

7. Perform each of the following operations.

   a. $\phi \cap \{\phi\}$

   b. $\{\phi, \{\phi\}\} - \{\{\phi\}\}$

   c. $\{\phi, \{\phi\}\} - \{\phi\}$

   d. $\{\{\{\phi\}\}\} - \phi$

8. Let $U = \{2, 3, 6, 8, 9, 11, 13, 15\}$,

   $A = \{x | x$ is a positive prime factor of $66\}$

   $B = \{ x \in U | \ x$ is composite number $\}$ and $C = \{x \in U | \ x - 5 \in U\}$. Then find each of the following.

$A \cap B, (A \cup B) \cap C, (A - B) \cup C, (A - B) - C, A - (B - C), (A - C) - (B - A), A' \cap B' \cap C'$

9. Let $A \cup B = \{a, b, c, d, e, x, y, z\}$ and $A \cap B = \{b, e, y\}$.

   a. $If\ B - A = \{x, z\}$, then $A =$ ___________________

   b. $If\ A - B = \phi$, then $B =$ ____________________

   c. $If\ B = \{b, e, y, z\}$, then $A - B =$ ________________

10. Let $U = \{1, 2, \ldots, 10\}, A = \{3, 5, 6, 8, 10\}, B = \{1, 2, 4, 5, 8, 9\}$,

    $C = \{1, 2, 3, 4, 5, 6, 8\}$ and $D = \{2, 3, 5, 7, 8, 9\}$. Verify each of the following.

    a. $(A \cup B) \cup C = A \cup (B \cup C)$.

    b. $A \cap (B \cup C \cup D) = (A \cap B) \cup (A \cap C) \cup (A \cap D)$.

    c. $(A \cap B \cap C \cap D)' = A' \cup B' \cup C' \cup D'$.

    d. $C - D = C \cap D'$.

    e. $A \cap (B \cap C)' = (A - B) \cup (A - C)$.

11. Depending on question No. 10 find.

a. $A \Delta B$.

b. $C \Delta D$.

c. $(A \Delta C) \Delta D$.

d. $(A \cup B) \setminus (A \Delta B)$.

12. For any two subsets A and B of a universal set U, prove that:

a. $A \Delta B = B \Delta A$.

b. $A \Delta B = (A \cup B) - (A \cap B)$.

c. $A \Delta \phi = A$.

d. $A \Delta A = \phi$.

13. Draw an appropriate Venn diagram to depict each of the following sets.

a. *U* = The set of high school students in Addis Ababa.

A = The set of female high school students in Addis Ababa.

B = The set of high school anti-AIDS club member students in Addis Ababa.

C = The set of high school Nature Club member students in Addis Ababa.

b. *U* = The set of integers.

A = The set of even integers.

B = The set of odd integers.

C = The set of multiples of 3.

D = The set of prime numbers.

Chapter 2
The Real and Complex Number Systems

In everyday life, knowingly or unknowingly, we are doing with numbers. Therefore, it will be nice if we get familiarized with numbers. Whatever course (which needs the concept of mathematics) we take, we face with the concept of numbers directly or indirectly. For this purpose, numbers and their basic properties will be introduced under this chapter.

Objective of the Chapter

At the end of this chapter, students will be able to:

- check the closure property of a given set of numbers on some operations
- determine the GCF and LCM of natural numbers
- apply the principle of mathematical induction to prove different mathematical formulae
- determine whether a given real number is rational number or not
- plot complex numbers on the complex plane
- convert a complex number from rectangular form to polar form and vice-versa
- extract roots of complex numbers

2.1 The real number System

2.1.1 The set of natural numbers

The history of numbers indicated that the first set of numbers used by the ancient human beings for counting purpose was the set of natural (counting) numbers.

Definition 2.1.1

The set of natural numbers is denoted by **N** and is described as $\mathbf{N} = \{1, 2, 3, \ldots\}$

2.1.1.1 Operations on the set of natural numbers

i) Addition (+)

If two natural numbers a & b are added using the operation "+", then the sum $a+b$ is also a natural number. If the sum of the two natural numbers a & b is denoted by c, then we can write the operation as: $c = a+b$, where c is called the sum and a & b are called terms.

Example: 3+8 = 11, here 11 is the sum whereas 3 & 8 are terms.

ii) Multiplication (×)

If two natural numbers a & b are multiplied using the operation "×", then the product $a\times b$ is also a natural number. If the product of the two natural numbers a & b is denoted by c, then we can write the operation as: $c = a\times b$, where c is called the product and a & b are called factors.

Example 2.1.3: $3\times 4 = 12$, here 12 is the product whereas 3 & 4 are factors.

Properties of addition and multiplication on the set of natural numbers

i. For any two natural numbers a & b, the sum $a+b$ is also a natural number. For instance in the above example, 3 and 8 are natural numbers, their sum 11 is also a natural number. In general, we say that the set of natural numbers is closed under addition.

ii. For any two natural numbers a & b, $a + b = b + a$.

Example 2.1.1: 3+8 = 8+3 = 11. In general, we say that addition is commutative on the set of natural numbers.

iii. For any three natural numbers a, b & c, $(a+b)+c = a +(b+c)$.

Example 2.1.2: (3+8)+6 = 3+(8+6) = 17. In general, we say that addition is associative on the set of natural numbers.

iv. For any two natural numbers a & b, the product $a\times b$ is also a natural number. For instance in the above example, 3 and 4 are natural numbers, their product 12 is also a natural number. In general, we say that the set of natural numbers is closed under multiplication.

v. For any two natural numbers a & b, $a\times b = b\times a$.

Example 2.1.4: $3\times 4 = 4\times 3 = 12$. In general, we say that multiplication is commutative on the set of natural numbers.

vi. For any three natural numbers a, b & c, $(a\times b) \times c = a\times (b\times c)$.

Example 2.1.5: $(2\times 4) \times 5 = 2\times (4\times 5) = 40$. In general, we say that multiplication is associative on the set of natural numbers.

vi. For any natural number a, it holds that $a\times 1 = 1\times a = a$.

Example 2.1.6: $6\times 1 = 1\times 6 = 6$. In general, we say that multiplication has an identity element on the set of natural numbers and 1 is the identity element.

vii. For any three natural numbers a, b & c, $a\times (b+c) = (a\times b)+(a\times c)$.

Example 2.1.7: $3\times(5+7) = (3\times 5)+ (3\times 7) = 36$. In general, we say that multiplication is distributive over addition on the set of natural numbers.

Note: Consider two numbers a and b, we say a is greater than b denoted by $a>$b if $a - b$ is positive.

2.1.1.2 Order Relation in N

i) **Transitive property**:

For any three natural numbers a, b & c, $a > b$ & $b > c \Rightarrow a > c$

ii) **Addition property:**

For any three natural numbers a, b & c, $a > b \Rightarrow a + c > b + c$

iii) **Multiplication property**:

For any three natural numbers a, b and c, $a > b \Rightarrow ac > bc$

iv) **Law of trichotomy**

For any two natural numbers a & b we have $a > b$ *or* $a < b$ *or* $a = b$.

2.1.1.3 Factors of a number

Definition 2.2

If $a, b, c \in N$ such that $ab = c$, then *a* & *b* are factors (divisors) of *c* and *c* is called product (multiple) of *a* & *b*.

Example 2.8: Find the factors of 15.

Solution: Factors of 15 are 1, 3, 5, 15. *Or we can write it as*: $F_{15} = \{1, 3, 5, 15\}$

Definition 2.3 A number $a \in N$ is said to be

i. **Even** if it is divisible by 2.

ii. **Odd** if it is not divisible by 2.

iii. **Prime** if it has only two factors (1 and itself).

iv. **Composite**: if it has three or more factors.

Example 2.9: 2, 4, 6, . . . are even numbers

Example 2.10: 1, 3, 5, . . . are odd numbers

Example 2.11: 2, 3, 5, . . . are prime numbers

Example 2.12: 4, 6, 8, 9, . . . are composite numbers

Remark: 1 is neither prime nor composite.

2.1.1.5 Prime Factorization

Definition 2.4

Prime factorization of a composite number is the product of all its prime factors.

Example 2.9:

$a)\ 6 = 2 \times 3$ $b)\ 30 = 2 \times 3 \times 5$ $c)\ 12 = 2 \times 2 \times 3 = 2^2 \times 3$ $d)\ 8 = 2 \times 2 \times 2 = 2^3$ $e)\ 180 = 2^2 \times 3^2 \times 5$

Fundamental Theorem of Arithmetic:

Every composite number can be expressed as a product of its prime factors. This factorization is unique except the order of the factors.

2.1.1.6 Greatest Common Factor (GCF)

Definition 2.5

The greatest common factor (GCF) of two numbers *a* & *b* is denoted by GCF (*a*, *b*) and is the greatest number which is a factor of each of the given number.

Note: If the GCF of two numbers is 1, then the numbers are called relatively prime.

Example 2.10: Consider the two numbers 24 and 60.

$Now \;\; F_{24} = \{\, 1, 2, 3, 4, 6, 8, 12, 24 \,\}$
$and \;\; F_{60} = \{\, 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 \,\}$
$Next \; F_{24} \cap F_{60} = \{\, 1, 2, 3, 4, 6, 12 \,\} \;\; from \; which \;\; 12 \; is \; the \; greatest.$

Therefore, GCF(24, 60) = 12.

This method of finding the GCF of two or more numbers is usually lengthy and time consuming. Hence an alternative method (Prime factorization method) is provided as below:

Step 1: Find the prime factorization of each of the natural numbers

Step 2: Form the GCF of the given numbers as the product of every factor that appears in each of the prime factorization but take the least number of times it appears.

Example 2.11: Consider the two numbers 24 and 60.

$Step 1: \; 24 = 2^3 \times 3$

$60 = 2^2 \times 3 \times 5$

Step 2: The factors that appear in both cases are 2 and 3, but take the numbers with the least number of times.

$\therefore \;\; GCF\,(24, 60) = 2^2 \times 3 = \underline{\underline{12}}$

Example 2.12: Consider the three numbers 20, 80 and 450.

$Step 1: \; 20 = 2^2 \times 5$

$80 = 2^4 \times 5$

$450 = 2 \times 3^2 \times 5^2$

Step 2: The factors that appear in all cases are 2 and 5, but take the numbers with the least number of times.

$\therefore \;\; GCF\,(20, 80, 450) = 2 \times 5 = \underline{\underline{10}}$

2.1.1.7 Least Common Multiple (LCM)

Definition 2.6

The least common multiple (LCM) of two numbers *a* & *b* is denoted by LCM (*a*, *b*) and is the least number which is a multiple of each of the given number.

Example 2.13: Consider the two numbers 18 and 24.

$Now\ \ M_{18} = \{\ 18,\ 36,\ 54,\ 72, 90,\ 108,\ 126,\ 144,\ \cdots\ \}$
$and\ \ M_{24} = \{\ 24,\ 48,\ 72,\ 96, 120,\ 144,\ \cdots\ \}$
$Next\ \ M_{18} \cap M_{24} = \{\ 72,\ \ 144, \cdots\ \}\ from\ which\ \ 72\ is\ the\ least.$
Therefore, LCM (18, 24) = 72.

This method of finding the LCM of two or more numbers is usually lengthy and time consuming. Hence an alternative method (Prime factorization method) is provided as below:

Step 1: Find the prime factorization of each of the natural numbers

Step 2: Form the LCM of the given numbers as the product of every factor that appears in any of the prime factorization but take the highest number of times it appears.

Example 2.14: Consider the two numbers 18 and 24.

$Step 1:\ 18 = 2^2 \times 3^2$
$24 = 2^3 \times 3$

Step 2: The factors that appear in any case are 2 and 3, but take the numbers with the highest number of times.

$\therefore\ \ LCM\ (18,\ 24) = 2^3 \times 3^2\ = \underline{\underline{72}}$

Example 2.15: Consider the three numbers 20, 80 and 450.

$Step 1:\ 20 = 2^2 \times 5$
$80 = 2^4 \times 5$
$450 = 2 \times 3^2 \times 5^2$

Step 2: The factors that appear in any cases are 2 , 3 and 5, but take the numbers with the highest number of times.

$\therefore\ \ LCM\ \ (20,\ 80,\ 450) = 2^4 \times 3^2 \times 5^2\ = \underline{\underline{3600}}$

2.1.1.8 Well ordering Principle in the set of natural numbers

Proposition 2.7

Every non-empty subset of the set of natural numbers has smallest (least) element.

Example 2.16 $A = \{2,\ 3,\ 4,\ \cdots\ \} \subseteq N.\quad smallest\ element\ \ of\ \ A = 2.$

Note: The set of counting numbers including zero is called the set of whole numbers and is denoted by **W**. i.e **W** $= \{0,\ 1,\ 2, 3, \ldots\}$

2.1.1.9 Principle of Mathematical Induction

Mathematical induction is one of the most important techniques used to prove in mathematics. It is used to check conjectures about the outcome of processes that occur repeatedly according to definite patterns. We will introduce the technique with examples.

For a given assertion involving a natural number n, if

i. the assertion is true for n = 1 (usually).
ii. it is true for n = k+1, whenever it is true for n = k ($k \geq 1$), then the assertion is true for every natural number n.

The method is used to prove different propositions involving positive integers using three steps:

Step1: Prove that T_k (usually T_1) holds true.

Step 2: Assume that T_k for k = n is true.

Step 3: Show that T_k is true for k = n+1.

Example 2.17 Show that $1+3+5+\cdots+(2n-1)=n^2$.

Proof:

$Step1.\ For\ n=1,\ 1=1^2\ which\ is\ true.$

$Step2.\ Assume\ that\ it\ is\ true\ for\ n=k$

$i.e.\ 1+3+5+\cdots+(2k-1)=k^2.$

$Step3.\ We\ should\ show\ that\ it\ is\ true\ for\ n=k+1.$

$Claim: 1+3+5+\cdots+(2k-1)+(2k+1)=(k+1)^2$

$$\begin{aligned} Now\ \underline{1+3+5+\cdots+(2k-1)}+(2k+1) &= k^2+(2k+1) \\ &= k^2+2k+1 \\ &= \underline{\underline{(k+1)^2}}\ which\ is\ the\ required\ result. \end{aligned}$$

$\therefore$ *It is true for any natural number n.*

Example 2.18 Show that $1+2+3+\cdots+(n)=\dfrac{n(n+1)}{2}$.

Proof:

$Step1.\ For\ n=1,\ 1=\dfrac{1(1+1)}{2}\ which\ is\ true.$

$Step2.\ Assume\ that\ it\ is\ true\ for\ n=k$

$i.e.\ 1+2+3+\cdots+(k)=\dfrac{k(k+1)}{2}.$

$Step3.\ We\ should\ show\ that\ it\ is\ true\ for\ n=k+1$

$$Claim: 1+2+3+\cdots+(k)+(k+1)=\frac{(k+1)(k+2)}{2}.$$

$$Now \quad \underline{1+2+3+\cdots+(k)}+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+1)(k+2)}{2} \quad \text{which is the required result.}$$

$\therefore$ *It is true for any natural number* n.

Example 2.19 Show that $5^n+6^n<9^n$ *for* $n\geq 2$.

Proof:

*Step*1. *For* $n=2$, $61<81$ *which is true*

*Step*2. *Assume that it is true for* $n=k$.

i.e. $5^k+6^k<9^k$.

*Step*3. *We should show that it is true for* $n=k+1$

Claim: $5^{k+1}+6^{k+1}<9^{k+1}$.

$$Now \quad 5^{k+1}+6^{k+1}=5.5^k+6.6^k<\underline{6}.5^k+6.6^k$$

$$=6(5^k+6^k)$$

$$<9(5^k+6^k)$$

$$<9(9^k)=9^{k+1}$$

$\Rightarrow 5^{k+1}+6^{k+1}<9^{k+1}$ *which is the required format.*

$\therefore$ *It is true for any natural number* $n\geq 2$.

2.1.2 The set of Integers

As the knowledge and interest of human beings increased, it was important and obligatory to extend the natural number system. For instance to solve the equation x+1= 0, the set of natural numbers was not sufficient. Hence the set of integers was developed to satisfy such extended demands.

Definition 2.8

The set of integers is denoted by **Z** and described as **Z** = $\{\ldots,-2,-1,0,1,2,\ldots\}$

2.1.2.1 Operations on the set of integers

i) Addition (+)

If two integers a & b are added using the operation "+", then the sum $a+b$ is also an integer. If the sum of the two integers a & b is denoted by c, then we can write the operation as: $c = a+b$, where c is called the sum and a & b are called terms.

Example 2.20: 4+9 = 13, here 13 is the sum whereas 4 & 9 are terms.

ii) Subtraction (−)

For any two integers a & b, the operation of subtracting b from a, denoted by $a-b$ is defined by $a-b=a+(-b)$. This means that subtracting b from a is equivalent to adding the additive inverse of b to a.

Example 2.21: $7-5=7+(-5)=2$

iii) Multiplication (×)

If two integers a & b are multiplied using the operation "×", then the product $a\times b$ is also an integer. If the product of the two integers a & b is denoted by c, then we can write the operation as: $c = a\times b$, where c is called the product and a & b are called factors.

Example 2.22: 4×7 = 28, here 28 is the product whereas 4 & 7 are factors.

Properties of addition and multiplication on the set of integers

i. For any two integers a & b, the sum $a+b$ is also an integer. For instance in the above example, 4 and 9 are integers, their sum 13 is also an integer. In general, we say that the set of integers is closed under addition.

ii. For any two integers a & b, $a+b = b+a$.

Example 2.23: 4+9 = 9+4 = 13. In general, we say that addition is commutative on the set of integers.

iii. For any three integers a, b & c, $(a+b)+c = a+(b+c)$.

Example 2.24: (5+9)+8 = 5+(9+8) = 22. In general, we say that addition is associative on the set of integers.

iv. For any integer a, it holds that $a+0 = 0+a = a$.

Example 2.25: 7+0 = 0+7 = 7. In general, we say that addition has an identity element on the set of integers and 0 is the identity element.

v. For any integer a, it holds that $a+(-a)=-a+a=0$.

Example 2.26: 4+-4 = -4+4 = 0. In general, we say that every integer a has an additive inverse denoted by $-a$.

vi. For any two integers a & b, the product $a\times b$ is also an integer. For instance in the above example, 4 and 7 are integers, their product 28 is also an integer. In general, we say that the set of integers is closed under multiplication.

vii. For any two integers a & b, $a\times b = b\times a$.

Example 2.27: $4\times7 = 7\times4 = 28$. In general, we say that multiplication is commutative on the set of integers.

viii. For any three integers a, b & c, $(a\times b)\times c = a\times(b\times c)$.

Example 2.28: $(3\times5)\times4 = 3\times(5\times4) = 60$. In general, we say that multiplication is associative on the set of integers.

ix. For any integer a, it holds that $a\times1 = 1\times a = a$.

Example 2.29: $5\times1 = 1\times5 = 5$. In general, we say that multiplication has an identity element on the set of integers and 1 is the identity element.

x. For any three integers a, b & c, $a\times(b+c) = (a\times b)+(a\times c)$.

Example 2.30: $4\times(5+6) = (4\times5)+(4\times6) = 44$. In general, we say that multiplication is distributive over addition on the set of integers.

2.1.2.2 Order Relation in Z

i) **Transitive property:** For any three integers a, b & c, $a > b$ & $b > c \Rightarrow a > c$

ii) **Addition property**: For any three integers a, b & c, $a > b \Rightarrow a+c > b+c$

iii) **Multiplication property**: For any three integers a, b and c, where c>0, $a > b \Rightarrow ac > bc$

iv) **Law of trichotomy**: For any two integers a & b we have $a > b$ or $a < b$ or $a = b$.

Exercise 2.1

1. Find an odd natural number x such that LCM (x, 40) = 1400.
2. There are between 50 and 60 number of eggs in a basket. When Loza counts by 3's, there are 2 eggs left over. When she counts by 5's, there are 4 left over. How many eggs are there in the basket?
3. The GCF of two numbers is 3 and their LCM is 180. If one of the numbers is 45, then find the second number.
4. Using Mathematical Induction, prove the following:

   a) $6^n - 1$ *is divisible by* 5, *for* $n \geq 0$.

   b) $2^n \leq (n+1)!$, *for* $n \geq 0$

   c) $x^n + y^n$ *is divisible by* $x + y$ *for odd natural number* $n \geq 1$.

   d) $2+4+6+\cdots+2n = n(n+1)$

   e) $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

f) $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

g) $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

2.1.3 The set of rational numbers

As the knowledge and interest of human beings increased with time, it was again necessary to extend the set of integers. For instance to solve the equation $2x+1=0$, the set of integers was not sufficient. Hence the set of rational numbers was developed to satisfy such extended needs.

Definition 2.9

Any number that can be expressed in the form $\dfrac{a}{b}$, where a and b are integers and $b \neq 0$, is called a rational number. The set of rational numbers denoted by Q is described by

$Q=\left\{\dfrac{a}{b} : a \text{ and } b \text{ are integers and } b \neq 0\right\}$.

Notes:

i. From the expression $\dfrac{a}{b}$, a is called numerator and b is called denominator.

ii. A rational number $\dfrac{a}{b}$ is said to be in lowest form if GCF $(a, b) = 1$.

2.1.3.1 Operations on the set of rational numbers

i) Addition (+)

If two rational numbers a/b and c/d are added using the operation "+", then the sum defined as $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$ is also a rational number.

Example 2.31: $\dfrac{1}{2}+\dfrac{3}{5}=\dfrac{11}{10}$

ii) Subtraction (−)

For any two rational numbers a/b & c/d, the operation of subtracting c/d from a/b, denoted by $a/b - c/d$ is defined by $a/b - c/d = a/b + (-c/d)$.

Example 2.32: $\dfrac{1}{2}-\dfrac{3}{5}=\dfrac{-1}{10}$

iii) Multiplication ($\times$)

If two rational numbers a/b and c/d are multiplied using the operation "$\times$", then the product defined as $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ is also a rational number.

Example 2.33: $\frac{1}{2}\times\frac{3}{5}=\frac{3}{10}$

iv) Division ($\div$)

For any two rational numbers $a/b \,\&\, c/d$, dividing a/b by c/d is defined by

$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c},\ c\neq 0.$

Example 2.34: $\frac{1}{2}\div\frac{3}{5}=\frac{1}{2}\times\frac{5}{3}=\frac{5}{6}$

Properties of addition and multiplication on the set of rational numbers

Let a/b, c/d and e/f be three rational numbers, then

i. The set of rational numbers is closed under addition and multiplication.

ii. Addition and multiplication are both commutative on the set of rational numbers.

iii. Addition and multiplication are both associative on the set of rational numbers.

iv. **0** is the additive identity

i.e., $a/b+\mathbf{0}=\mathbf{0}+a/b\ =\ a/b$.

v. Every rational number has an additive inverse.

i.e., $a/b+\ (-a/b)\ =\ -a/b+a/b\ =\mathbf{0}$.

vi. **1** is the multiplicative identity

i.e., $a/b\times 1=\mathbf{1}\times a/b\ =\ a/b$.

vii. Every non-zero rational number has a multiplicative inverse.

i.e., $a/b\times\ b/a\ =\ b/a\times a/b\ =1$.

2.1.3.2 Order Relation in Q

i) **Transitive property**

For any three rational numbers a/b, c/d & e/f $\ a/b>c/d$ & $c/d>e/f\ \Rightarrow a/b>e/f$.

ii) **Addition property**

For any three rational numbers a/b, c/d & e/f $\ a/b>c/d\ \Rightarrow a/b+e/f>c/d+e/f$.

iii) **Multiplication property**

For any three rational numbers a/b, c/d, e/f and $e/f>0$

$a/b > c/d \Rightarrow (a/b)(e/f) > (c/d)(e/f).$

iv) **Law of trichotomy**

For any two rational numbers a/b & c/d we have $a/b > c/d$ *or* $a/b < c/d$ *or* $a/b = c/d$.

2.1.3.3 Decimal representation of rational numbers

A rational number $\frac{a}{b}$ can be written in decimal form using long division.

2.1.3.3.1 Terminating decimals

Example 2.35: Express the fraction number $\frac{25}{4}$ in decimal form.

$Solution: \frac{25}{4} = 6.25$

2.1.3.3.2 Non-terminating periodic decimals

Example 2.36: Express the fraction number $\frac{25}{3}$ in decimal form.

$Solution: \frac{25}{3} = 8.333\cdots$

Now we will see how to convert decimal numbers in to their fraction forms. In earlier mathematics topics, we have seen that multiplying a decimal by 10 pushes the decimal point to the right by one position and in general, multiplying a decimal by 10^n pushes the decimal point to the right by n positions. We will use this fact for the succeeding topics.

2.1.3.4 Fraction form of decimal numbers

A rational number which is written in decimal form can be converted to a fraction form as $\frac{a}{b}$ in lowest (simplified) form, where a and b are relatively prime.

2.1.3.4.1 Terminating decimals

Consider any terminating decimal number d. Suppose d terminates n digits after the decimal point. d can be converted to its fraction form as below:

$$d = d \times 1 = d \times \frac{1}{1} = \underline{\underline{d \times (\frac{10^n}{10^n})}}.$$

Example 2.37: Convert the terminating decimal 3.47 to fraction form.

$$Solution: 3.47 = 3.47 \times \frac{10^2}{10^2} = \underline{\underline{\frac{347}{100}}}.$$

2.1.3.4.2 Non-terminating periodic decimals

Consider any non-terminating periodic decimal number d. Suppose d has **k** non-terminating digits and **p** terminating digits after the decimal point. d can be converted to its fraction form as below:

$$d = d\times 1 = d\times\frac{1}{1} = \underline{\underline{d\times(\frac{10^{k+p}-10^{k}}{10^{k+p}-10^{k}})}}.$$

Example 2.38: Convert the non-terminating periodic decimal $42.5\overline{38}$ to fraction form.
Solution: k = 1, p = 2.

$$\therefore\ d = d\times 1 = d\times\frac{1}{1} = d\times(\frac{10^{k+p}-10^{k}}{10^{k+p}-10^{k}}) = 42.5\overline{38}\times(\frac{10^{3}-10}{10^{3}-10}) = \frac{42538.\overline{38}-425.\overline{38}}{1000-10} = \underline{\underline{\frac{42113}{990}}}.$$

Note: From the above two cases, we can conclude that both terminating decimals and non-terminating periodic decimals are rational numbers. (Why? Justify).

2.1.3.5 Non-terminating and non-periodic decimals

Some decimal numbers are neither terminating nor non-terminating periodic. Such types of numbers are called irrational numbers.

Example 2.39: 62.757757775….

Example 2.40: Show that $\sqrt{2}$ is an irrational number.

Proof:

$$\textit{Suppose } \sqrt{2} \textit{ is a rational number}$$
$$\Rightarrow \sqrt{2} = \frac{a}{b},\ \textit{where } GCF(a,b)=1$$
$$\Rightarrow 2 = \frac{a^2}{b^2}$$
$$\Rightarrow a^2 = 2b^2 ....(*)$$
$$\Rightarrow a^2 \textit{ is even}$$
$$\Rightarrow a \textit{ is even}$$
$$\Rightarrow a = 2n \text{ ....}(**)$$

$$\textit{Putting this in } (*) \textit{ we get}:$$
$$\Rightarrow 4n^2 = 2b^2$$
$$\Rightarrow b^2 = 2n^2$$
$$\Rightarrow b^2 \textit{ is even}$$
$$\Rightarrow b \textit{ is even}$$
$$\Rightarrow b = 2m ....(***)$$

From (**) and (***) we get a contradiction that GCF (a, b) = 1 which implies that $\sqrt{2}$ is not a rational number.

Therefore, $\sqrt{2}$ is an irrational number.

2.1.4 The set of real numbers

Definition 2.10
A number is called a real number if and only if it is either a rational number or an irrational number. The set of real numbers denoted by $\Re$ can be described as the union of the set of rational and irrational numbers. i.e $\Re = \{x : x \text{ is a rational number or an irrational number}\}$.

There is a 1-1 correspondence between the set of real numbers and the number line (For each point in the number line, there is a corresponding real number and vice-versa).

2.1.4.1 Operations on the set of real numbers

i) Addition (+)

If two real numbers are added using the operation "+", then the sum is also a real number.

ii) Subtraction ($-$)

For any two real numbers a & b, the operation of subtracting b from a, denoted by $a-b$ is defined by $a-b = a+(-b)$.

iii) Multiplication ($\times$)

If two real numbers a and b are multiplied using the operation "$\times$", then the product defined as $a \times b = ab$ is also a real number.

iv) Division ($\div$)

For any two real numbers a & b, dividing a by b is defined by $a \div b = a \times \frac{1}{b}$, $b \neq 0$.

Properties of addition and multiplication on the set of real numbers

Let a, b and c be three real numbers, then

i. The set of real numbers is closed under addition and multiplication.

ii. Addition and multiplication are commutative on the set of real numbers.

iii. Addition and multiplication are associative on the set of real numbers.

iv. **0** is the additive identity

i.e., $a + \mathbf{0} = \mathbf{0} + a = a$.

v. Every real number has an additive inverse.

i.e., $a + (-a) = -a + a = \mathbf{0}$.

vi. **1** is the multiplicative identity

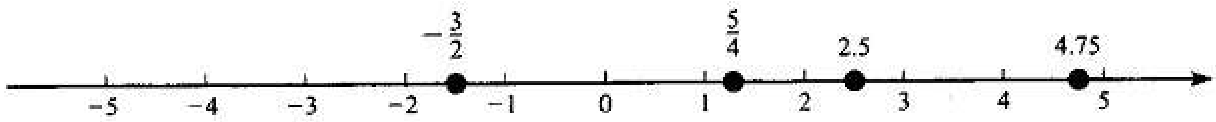
i.e., $a \times 1 = \mathbf{1} \times a = a$.

vii. Every non-zero real number has a multiplicative inverse.

i.e., $a \times 1/a = 1/a \times a = 1$.

2.1.4.2 The real number and the number line

One of the most important properties of the real number is that it can be represented graphically by points on a straight line. The point 0 is termed as the origin. Points to the right of 0 are called positive real numbers and points to the left of 0 are called negative real numbers. Each point on the number line corresponds a unique real number and vice-versa.



Geometrically we say a is greater than b if a is located to the right of b on the number line.

2.1.4.3 Order Relation in R

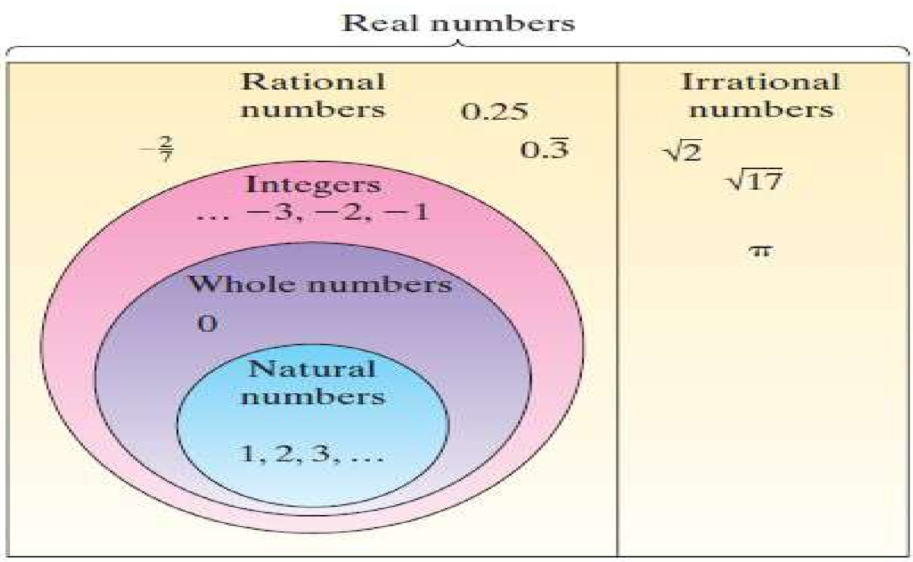
i) **Transitive property:** For any three real numbers a, b & c, $a > b$ & $b > c \Rightarrow a > c$.

ii) **Addition property:** For any three real numbers a, b & c, $a > b \Rightarrow a + c > b + c$.

iii) **Multiplication property:** For any three real numbers a, b, c and $c > 0$, we have $a > b \Rightarrow ac > bc$.

iv) **Law of trichotomy:** For any two real numbers a & b we have $a > b$ or $a < b$ or $a = b$.

Summary of the real number system



2.1.4.4 Intervals

Let a and b be two real numbers such that $a<b$, then the intervals which are subsets of **R** with end points a and b are denoted and defined as below:

i. $(a,b)=\{\ x: a<x<b\}$ open interval from a to b.

ii. $[a,b]=\{\ x: a\le x\le b\}$ closed interval from a to b.

iii. $(a,b]=\{\ x: a<x\le b\}$ open-closed interval from a to b.

iv. $[a,b)=\{\ x: a\le x<b\}$ closed-open interval from a to b.

2.1.4.5 Upper bounds and lower bounds

Definition 2.11

$Let\ A\ be\ non-empty\ and\ A\subseteq \Re.$

i. A point $a\in R$ is said to be an upper bound of A iff $x\le a$ for all $x\in A.$

ii. An upper bound of A is said to be least upper bound (lub) iff it is the least of all upper bounds.

iii. A point $a\in R$ is said to be lower bound of A iff $x\ge a$ for all $x\in A.$

ii. A lower bound of A is said to be greatest lower bound (glb) iff it is the greatest of all lower bounds.

Example 2.41 *Consider the set* $A=[2,\ 5)\subseteq \Re.$

$i)$ *lower bounds are* $\cdots, -9, -3, 0, \frac{1}{2}, 1, 2$

Here the greatest element is 2.

$\therefore\ \text{glb}=2$

$ii)$ *upper bounds are* $5, 6, \frac{25}{3}, 20, 99, 1000\cdots$

Here the least element is 5.

$\therefore\ \text{lub}=5.$

Example 2.42: Consider the set **A =** $\left\{\frac{1}{n}\right\}$ for n$\in N.$

Solution: $A=\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$

$i)$ *lower bounds are* $\cdots, -3, -2, 0$

Here the greatest element is $0.$ *Thus*, $\text{glb}=0$

$ii)$ *upper bounds are* $1, 3, \frac{9}{2}, 50, \cdots$

Here the least element is $1.$ *Thus*, $\text{lub}=1.$

Based on the above definitions, we can define the completeness property of real numbers as below.

2.1.4.4 Completeness property of real number (R)

Completeness property of real numbers states that: Every non-empty subset of $\Re$ that has lower bounds has glb and every non-empty subset of $\Re$ that has upper bounds has a lub.

Exercise 2.2

1. Express each of the following rational numbers as decimal:

   $a)\ \frac{4}{9}$ $b)\ \frac{3}{25}$ $c)\ \frac{11}{7}$ $d)\ -5\frac{2}{3}$ $e)\ \frac{2}{77}$

2. Write each of the following as decimal and then as a fraction:

   a) three tenths b) four thousands

3. Write each of the following in meters as a fraction and then as a decimal

   a) 4mm b) 6cm and 4mm c) 56cm and 4mm

4. Classify each of the following as terminating or non-terminating periodic

   $a)\ \frac{5}{13}$ $b)\ \frac{7}{10}$ $c)\ \frac{69}{64}$ $d)\ \frac{11}{60}$ $e)\ \frac{5}{12}$

5. Convert the following decimals to fractions:

   $a)\ 3.2\overline{5}$ $b)\ 0.3\overline{14}$ $c)\ 0.\overline{275}$

6. Determine whether the following are rational or irrational:

   $a)\ 2.7\overline{5}$ $b)\ 0.272727\cdots$ $c)\ \sqrt{8}-\frac{1}{\sqrt{2}}$

7. Which of the following statements are true and which of them are false?

a) The sum of any two rational numbers is rational

b) The sum of any two irrational numbers is irrational

c) The product of any two rational numbers is rational

d) The product of any two irrational numbers is irrational

11. Find two rational numbers between $\frac{1}{3}$ *and* $\frac{1}{2}$.

2.2 The set of complex numbers

The positive integers (natural numbers) were invented to count things. The negative integers were introduced to count money when we owed more than we had. The rational numbers were invented for measuring quantities. Since quantities like voltage, length and time can be measured using fractions, they can be measured using the rational numbers.

The real numbers were invented for wholly mathematical reasons: it was found that there were lengths such as the diagonal of the unit square which, in principle, couldn't be measured by the rational numbers, instead they can be measured using real numbers.

The complex numbers were invented for purely mathematical reasons, just like the real numbers and were intended to make things neat and tidy in solving equations. They were regarded with deep suspicion by the more conservative folk for a century. Complex numbers are points in the plane, together with a rule telling you how to multiply them. They are two-dimensional, whereas the real numbers are one dimensional.

Equations of the form $x^2+1=0$ has no solution on the set of real numbers. Therefore, the set of complex numbers permits us to solve such equations.

Definition 2.12

The set of complex numbers is denoted by ℂ and is described by
$\mathbb{C}=\left\{z/z=x+iy,\ x,\ y\in\Re\ \ and\ \ i^2=-1\right\}.$

From the expression $z=x+iy$, *x is called the real part and is denoted by* $\mathrm{Re}(Z)$

y is called the imaginary part and is denoted by $\mathrm{Im}(Z)$.

Note: If x = 0, the number is called purely imaginary and if y = 0, the number is called purely real.
Complex numbers can be defined as an order pair (x, y) of real numbers that can be interpreted as points in the complex plane (z- plane) with coordinates x and y.

Example 2.43: *Find the real & imaginary part of the following complex numbers :*

$a)\ z=3+7i$

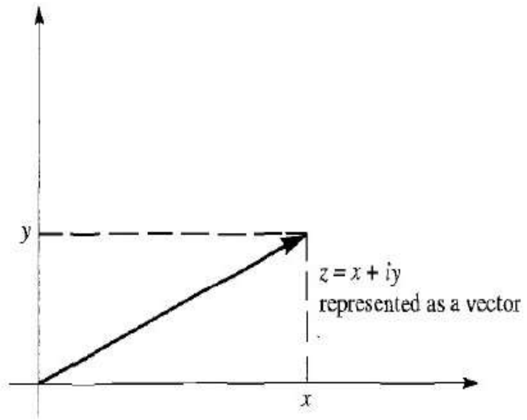
Solution : *real part* $=3$ & *imaginary part* $=7$

$b)\ z=1-i$

Solution : *real part* $=1$ & *imaginary part* $=-1$

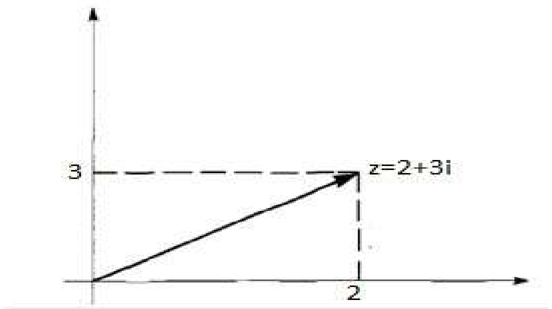
2.2.1 Plotting complex numbers

Any complex number $z = x + iy$ *can be drawn in the complex plane as below*:



Example 2.44: Draw the complex number z = 2+3i

Solution:



Equality of Complex numbers

Two complex numbers $z_1 = a + ib$ *and* $z_2 = c + id$ *are equal iff* $a = c$ & $b = d$.

Example 2.45 *If* $z_1 = 2 + ix$ *and* $z_2 = y + 6i$ *are equal, then find the value of* x & y.

Solution:

$x = 6, \; y = 2.$

2.2.2 Operations on Complex numbers

Let $z_1 = a + ib$ *and* $z_2 = c + id$ *be any two complex numbers, then*

i) $z_1 + z_2 = (a + c) + i(b + d)$

ii) $z_1 - z_2 = (a - c) + i(b - d)$

iii) $z_1 . z_2 = (a + ib).\ (c + id) = a(c + id) + ib(c + id) = ac + iad + ibc - bd = (ac - bd) + i(ad + bc)$

iv) $\dfrac{z_1}{z_2} = \dfrac{(a + ib)}{(c + id)}, \; z_2 \neq 0.$

Example 2.46 *If* $z_1 = 2+3i$ *and* $z_2 = 4+i$, *then find* $a)\ z_1+z_2 \quad b)\ z_1-z_2 \quad c)\ z_1.z_2 \quad d)\ \frac{z_1}{z_2}$

$Sol: a)\ z_1+z_2 = 6+4i$

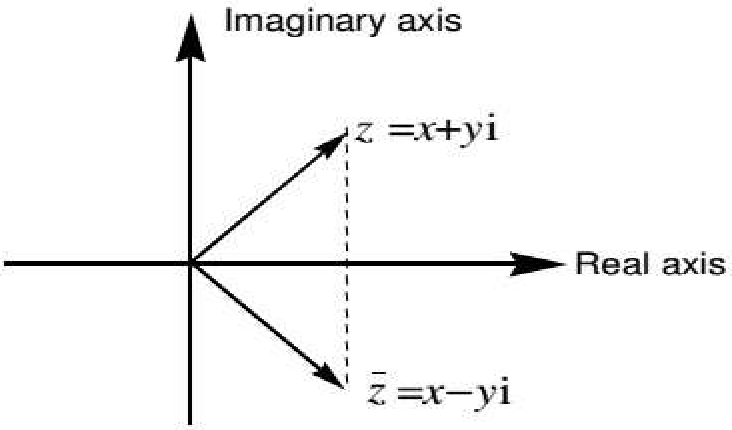
$b)\ z_1-z_2 = -2+2i$

$c)\ z_1.z_2 = (2+3i).(4+i) = 8+2i+12i-3 = 5+14i$

$d)\ \frac{z_1}{z_2} = \frac{2+3i}{4+i}$

2.2.3 Conjugate of a complex number

Definition 2.13
The conjugate of a complex number z = x+iy is denoted by $\bar{z}$ and is defined as $\bar{z}$ = x-iy. It can be represented by the point (x, -y) which is the reflection of the point (x, y) about the x-axis.



Example 2.47: Find the conjugate of the complex number z = 2+9i.

Solution :

$z = 2+9i$

$\Rightarrow \bar{z} = 2-9i$

Properties of Conjugate

$a.\ \bar{\bar{z}} = z$

$b.\ z+\bar{z} = 2x = 2\,\mathrm{Re}(z) = 2\left(\frac{z+\bar{z}}{2}\right)$

$c.\ z-\bar{z} = 2iy = 2i\,\mathrm{Im}(z)$

$d.\ \overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$

$e.\ \overline{z_1-z_2} = \overline{z_1}-\overline{z_2}$

$f.\ \overline{z_1.z_2} = \overline{z_1}.\overline{z_2}$

$g.\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

$proof: \ a) \ let \ z = x + iy$

$\Rightarrow \bar{z} = x - iy$

$\Rightarrow \bar{\bar{z}} = x + iy$

$\therefore \quad \bar{\bar{z}} = z.$

$d) \ Let \ z_1 = x_1 + iy_1 \ \& z_2 = x_2 + iy_2$

$\Rightarrow \overline{z_1} = x_1 - iy_1 \quad \& \overline{z_2} = x_2 - iy_2$

$$Now \ \overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2)$$

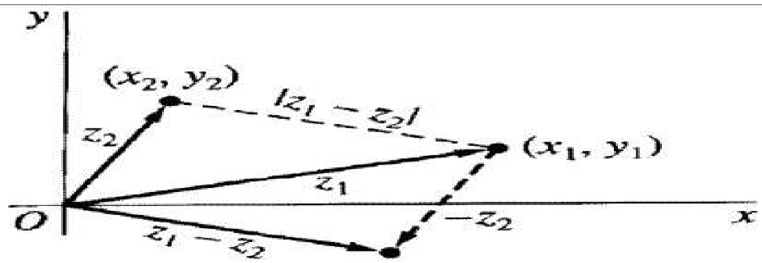
$$= x_1 - iy_1 + x_2 - iy_2$$

$$= \overline{z_1} + \overline{z_2}$$

The others are left for the reader.

2.2.4 Modulus (Norm) of a complex number

Definition 2.14
The modulus of a complex number z = x+iy is a non-negative real number denoted by |z| and is defined as |z| = $\sqrt{x^2 + y^2}$. Geometrically, the number |z| represents the distnce between the point (x, y) and the origin.



Example 2.48: Find the modulus of the complex number z = 3 – 4i.

$Solution: \ z = 3 - 4i$

$$|z| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

Properties of modulus

$a. \ |z| = |\bar{z}|$ $\qquad$ $b. \ |z|^2 = z.\bar{z}$ $\qquad$ $c. \ |z_1 . z_2| = |z_1| . |z_2|$

$d. \ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $\qquad$ $e. \ |z_1 + z_2| \le |z_1| + |z_2|$.....................*triangle inequality*

$f. \ |z_1 - z_2| \ge |z_1| - |z_2|$

$proof\ (a)\ let\ z = x + iy \quad from \quad which \quad \bar{z} = x - iy$

$$\Rightarrow |z| = \sqrt{x^2 + y^2} \qquad \Rightarrow |\bar{z}| = \sqrt{x^2 + y^2}$$

$$\therefore \quad |z| = |\bar{z}|.$$

$proof\ (b)\ let\ z = x + iy \quad from \quad which \quad \bar{z} = x - iy$

$$\Rightarrow |z|^2 = x^2 + y^2$$

$Now \quad z.\bar{z} = (x + iy).(x - iy) = x^2 + y^2$

$$\therefore \quad |z|^2 = z.\bar{z}$$

$$proof\ (c) \quad |z_1.z_2|^2 = (z_1.z_2)(\overline{z_1.z_2}) = (z_1.z_2)(\overline{z_1}.\overline{z_2}) = z_1.\overline{z_1}.z_2.\overline{z_2}$$

$$= |z_1|^2.|z_2|^2$$

$$= (|z_1|.|z_2|)^2 \text{..........} from\ (b)$$

$$\Rightarrow |z_1.z_2| = |z_1|.|z_2|$$

$$proof \quad (d) \quad \left|\frac{z_1}{z_2}\right|^2 = \frac{z_1(\overline{z_1})}{z_2(\overline{z_2})} = \frac{|z_1|^2}{|z_2|^2} = \left(\frac{|z_1|}{|z_2|}\right)^2$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

The others are left for the reader.

2.2.5 Additive and multiplicative inverses

Let z = x+iy be a complex number, then

$i)\ its\ additive\ inverse\ denoted\ by\ (-z)\ is\ given\ by: -z = -(x + iy) = -x - iy.$

$ii)\ its\ multiplicative\ inverse\ denoted\ by\ z^{-1}\ is\ given\ by: z^{-1} = \dfrac{1}{x + iy} = \dfrac{x}{x^2 + y^2} - \dfrac{iy}{x^2 + y^2}.$

Example 2.49: Find the additive and the multiplicative inverse of z = 3+4i.

$Solution: z = 3 + 4i$

$i) \quad -z = -3 - 4i$

$$ii)\ z^{-1} = \frac{1}{3 + 4i} = \frac{1}{3 + 4i}.\frac{3 - 4i}{3 - 4i} = \frac{3}{25} - \frac{4i}{25}.$$

Exercise 2.3

1. Verify that

$a)\ (\sqrt{2} - i) - i(1 - \sqrt{2}\,i) = -2i \qquad b)\ (2, -3)\ (-2, 1) = (-1, 8)$

$c)\ (3, 1)\ (3, -1)\ (\frac{1}{5}, \frac{1}{10}) = (2, 1) \qquad d)\ (2 + 3i)^2 - (3i - 6) = 1 + 9i$

2. Show that

   $a)\ \operatorname{Re}(iz) = -\operatorname{Im}(z)$ $\quad b)\ \operatorname{Im}(iz) = \operatorname{Re}(z)$ $\quad c)\ (z+1)^2 = z^2 + 2z + 1$

3. Do the following operations and simplify your answer.

   $a)\ \dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$ $\quad b)\ \dfrac{5i}{(1-i)(2-i)(3-i)}$ $\quad c)\ (1-i)^{3}$

4. Locate the complex numbers z_1+z_2 and z_1-z_2, as vectors where

   $a)\ z_1 = 2i\ ,\ z_2 = \dfrac{2}{3} - i$ $\quad b)\ z_1 = (-\sqrt{3}, 1)\ ,\ z_2 = (\sqrt{3}, 0)$

   $c)\ z_1 = (-3, 1)\ ,\ z_2 = (1, 4)$ $\quad d)\ z_1 = a + ib\ ,\ z_2 = a - ib$

5. Sketch the following set of points determined by the condition given below:

   $a)\ |z - 1 + i| = 1$ $\quad b)\ |z + i| \le 3$ $\quad c)\ |z - 4i| \ge 4$

6. Using properties of conjugate and modulus, show that

   $a)\ \overline{\bar{z} + 3i} = z - 3i$ $\quad b)\ \overline{iz} = -i\,\bar{z}$ $\quad c)\ \overline{(2+i)^2} = 3 - 4i$

7. Show that $(-1+i)^7 = 8(-1-i)$.

8. Using mathematical induction, show that (when n = 2, 3, . . . ,)

   $a)\ \overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$ $\quad b)\ \overline{z_1 z_2 \cdots z_n} = \bar{z}_1\ \bar{z}_2 \cdots \bar{z}_n$

9. Show that the equation $|z - z_0| = r$ which is a circle of radius r centered at z_0 can be written as $|z|^2 - 2\operatorname{Re}(z\bar{z}) + |z_0|^2 = r^2$.

2.2.6 Argument (Amplitude) of a complex number

Definition 2.15

Argument of a complex number z = x+iy is the angle formed by the complex number z = x+iy with the positive x-axis. The argument of a complex number z = x+iy is deonted by **argz** and is given by arg(z) = $\tan^{-1}(y/x)$.

The particular argument of z that lies in the range $-\pi < \theta \le \pi$ is called the principal argument of z and is dented by Argz.

Notes :i) $Argz \in (-\pi, \pi]$.

ii) If $0 \le Argz \le \pi$, *move counter clock wise direction, if not move the other direction.*

Example 2.50: Find the principal argument of the following complex numbers:

$a)\ z = 1+i \qquad b)\ z = -2+2\sqrt{3}i \qquad c)\ z = -\sqrt{3}-i$

$Sol:\quad a)\ z = 1+i$

$$Argz = \tan^{-1}(1/1) = \tan^{-1}(1) = \frac{\pi}{4}$$

$b)\ z = -2+2\sqrt{3}\,i$

$$Argz = \tan^{-1}(2\sqrt{3}/-2) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

$c)\ z = -\sqrt{3}-i$

$$Argz = \tan^{-1}(-1/-\sqrt{3}) = \tan^{-1}(1/\sqrt{3}) = \frac{-5\pi}{6}$$

Properties of Arguments

$i)\ Arg(z_1.z_2) = Argz_1 + Argz_2 \qquad ii)\ Arg(\frac{z_1}{z_2}) = Argz_1 - \arg z_2$

Example 2.51: Find the principal argument of $a)\ (1+i)(-1-i) \quad b)\ (\frac{-2+2i}{1-i})$

Solution

$$a)\ Arg(1+i)(-1-i) = Arg(1+i) + Arg(-1-i) = \pi/4 + (-3\pi/4) = -\pi/2$$

$$b)\ Arg(\frac{-2+2i}{1-i}) = Arg(-2+2i) - Arg(1-i) = 3\pi/4 - (-\pi/4) = 4\pi/4 = \pi$$

2.2.7 Polar form of a complex number

Definition 2.16
Let r and θ be polar coordinates of the point (x, y) of the complex number z = x+iy. Since x = $r\cos\theta$ and y = $r\sin\theta$, then the complex number can be written as : $z = r(\cos\theta + i\sin\theta)$ which is called polar form, where r is modulus of z and θ is principal argument of z.

Example 2.52: Express the following complex numbers in polar form:

$a)\ z = 1+i$

$solution: r = \sqrt{2}\ and\ \theta = \tan^{-1}(1) = \pi/4.\ Thus,\ z = \sqrt{2}(\cos\pi/4 + i\sin\pi/4).$

$b)\ z = 3 - 3i$

$solution:\ r = \sqrt{18}\ and\ \theta = \tan^{-1}(-1) = -\pi/4.$

$Thus,\ z = \sqrt{18}(\cos -\pi/4 + i\sin -\pi/4) = \sqrt{18}(\cos \pi/4 - i\sin \pi/4).$

Multiplication and division in polar forms

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, *then*

$a)\ z_1.z_2 = r_1.r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$ $\quad b)\ \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)).$

Proof:

$$\begin{aligned} a)\ z_1.z_2 &= r_1(\cos\theta_1 + i\sin\theta_1).r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1.r_2[\cos\theta_1(\cos\theta_2 + i\sin\theta_2) + i\sin\theta_1(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1.r_2[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2] \\ &= r_1.r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= \underline{\underline{r_1.r_2[\cos(\theta_1+\theta_2) + i\ \sin(\theta_1+\theta_2)]}} \end{aligned}$$

$$\begin{aligned} b)\ \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\left(\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right) \\ &= \frac{r_1}{r_2}\left(\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right).\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2} \\ &= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right) \\ &= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i\ (\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{1}\right) \\ &= \underline{\underline{\frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]}} \end{aligned}$$

Example 2.53: *If* $z_1 = 6(\cos \pi/2 + i\sin \pi/2)$ *and* $z_2 = 2(\cos \pi/3 + i\sin \pi/3)$, *then find*

$a)\ z_1.z_2 \qquad b)\ \dfrac{z_1}{z_2}$

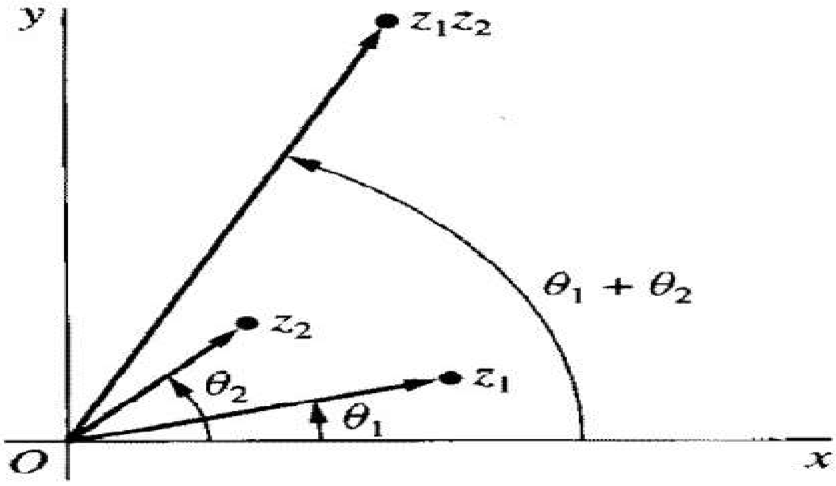
Solution:

$$\begin{aligned} a)\ z_1.\ z_2 &= 6.\,2[\cos(\pi/2 + \pi/3) + i\sin(\pi/2 + \pi/3)] \\ &= \underline{\underline{12\ [\cos 5\pi/6 + i\sin 5\pi/6]}} \end{aligned}$$

$$\begin{aligned} b)\ \frac{z_1}{z_2} &= \frac{6}{2}[\cos(\pi/2 - \pi/3) + i\sin(\pi/2 - \pi/3)] \\ &= \underline{\underline{3[\cos \pi/6 + i\sin\cos \pi/6]}} \end{aligned}$$

- **Argument of a product**

The argument of the product of two complex numbers is the sum of their arguments.



Proof:

$$Let\ z_1 = r_1(\cos\theta_1 + i\sin\theta_1)\ and\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

$$Now\ z_1.z_2 = r_1(\cos\theta_1 + i\sin\theta_1).r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1.r_2[\cos\theta_1(\cos\theta_2 + i\sin\theta_2) + i\sin\theta_1(\cos\theta_2 + i\sin\theta_2)]$$

$$= r_1.r_2[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2]$$

$$= r_1.r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$$

$$= \underline{\underline{r_1.r_2[\cos(\theta_1 + \theta_2) + i\ \sin(\theta_1 + \theta_2)]}}$$

$$\Rightarrow \arg(z_1.z_2) = \tan^{-1}\left(\frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)}\right)$$
$$= \tan^{-1}(\tan(\theta_1 + \theta_2))$$

$$= \underline{\underline{\theta_1 + \theta_2}}$$

- **Argument of a quotient**

The argument of the quotient of two complex numbers is the difference of their arguments.

Proof:

$$\arg(\frac{z_1}{z_2}) = \arg(z_1.z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1}) = \arg(z_1) + -(\arg z_2) = \arg(z_1) - \arg(z_2)$$

Example 2.54: $\arg(\frac{-4}{1+\sqrt{3}\,i}) = \arg(-4) - \arg(1+\sqrt{3}\,i) = \pi - \pi/3 = 2\pi/3$

De Moivre's Formula

recall the product: $z_1.z_2 = r_1.r_2\,[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$

Similarly, we get $z_1.z_2.\cdots.z_n = r_1.r_2.\cdots.r_n\,[\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)].$

Now we can generalize that $z^n = r^n[\cos(\theta+\theta+\cdots+\theta) + i\sin(\theta+\theta+\cdots+\theta)]$

$= \underline{\underline{r^n[\cos n\theta + i\sin n\theta]}}$ *which is called De Moivre's formula.*

Example 2.55: Express $(2+2i)^{100}$ in polar form.

Solution: *Let* $z = 2+2i$. *Then,* $r = \sqrt{8}$, $\theta = \pi/4$ *and hence*

$$(2+2i)^{100} = z^{100}$$
$$= \sqrt{8}^{100}\,[\cos 100(\pi/4) + i\sin 100(\pi/4)]$$
$$= \underline{\underline{8^{50}[\cos 25\pi + i\sin 25\pi]}}.$$

Example 2.56: Express $(\sqrt{3}+i)^{60}$ in polar form.

Solution: *Let* $z = \sqrt{3}+i$. *Then,* $r = 2$, $\theta = \pi/6$.

$$\therefore\ (\sqrt{3}+i)^{60} = z^{60}$$
$$= 2^{60}\,[\cos 60(\pi/6) + i\sin 60(\pi/6)]$$
$$= \underline{\underline{2^{60}[\cos 10\pi + i\sin 10\pi]}}.$$

Euler's formula

The complex number $z = r(\cos\theta + i\sin\theta)$ can be written in exponential form as: $z = re^{i\theta}$ which is called Euler's formula.

Note: $z^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{i(n\theta)}$

Example 2.57: Express the complex number z = 1+i using Euler's formula.

Solution: $z = 1+i$

Now $r = \sqrt{2}$ & $\theta = \pi/4 \quad \Rightarrow z = re^{i\theta} = \underline{\underline{\sqrt{2}e^{\pi i/4}}}$

Example 2.58: Express the complex number $z = 1+\sqrt{3}\,i$ using Euler's formula.

$Solution: z = 1 + \sqrt{3}\, i$

$Now \quad r = 2 \quad \& \quad \theta = \pi/3 \quad \Rightarrow \quad z = re^{i\theta} = \underline{\underline{2e^{\pi i/3}}}$

Example 2.59: Express the complex number $z = (\sqrt{3} + i)^7$ using Euler's formula.

$Solution: z = (\sqrt{3} + i)^7$

$Now \quad r = 2, \ \theta = \pi/6 \quad \Rightarrow (\sqrt{3} + 1)^7 = 2^7 e^{\frac{7\pi i}{6}} = 128 e^{\frac{7\pi i}{6}}$

2.2.8 Extraction of roots

Suppose $z_o = r_o e^{i\theta_o}$ *is the* n^{th} *root of a non-zero complex number* $z = re^{i\theta}$, *where* $n \geq 2$.

Then, $z_o^{\ n} = z$, *which implies that* $r_o^{\ n} e^{in\theta_o} = re^{i\theta}$

$$\Rightarrow r_o^{\ n} = r \quad \& \quad n\theta_o = \theta + 2k\pi, \quad k = 0, 1, 2, \cdots (n-1).$$

$$\Rightarrow r_o = (r)^{1/n} \quad \& \quad \theta_o = \frac{\theta}{n} + \frac{2k\pi}{n}$$

$\therefore \quad z_o = \underline{\underline{(r)^{1/n} (e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})})}}$ *which is the* n^{th} *root of* z, *where* $n = 2, 3, \cdots$ *and* $k = 0, 1, 2, \cdots (n-1)$

or we can denote it by C_k *as :* $C_k = \underline{\underline{(r)^{1/n} (e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})})}}$, $\quad k = 0, 1, 2, \cdots (n-1)$

Example 2.60: Find the square roots of the complex number $z = 1 + \sqrt{3} i.$

Solution :

$z = 1 + \sqrt{3}\, i$

$Here \quad r = 2, \ \theta = \pi/3$

$Hence \quad C_k = (r)^{1/n} (e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})}), \ n = 2 \ , \ k = 0, 1.$

$$\Rightarrow C_k = (2)^{1/2} (e^{i(\frac{\pi/3}{2} + \frac{2k\pi}{2})})$$

$$\Rightarrow C_k = \sqrt{2} (e^{i(\pi/6 + k\pi)})$$

$i.) \ If \ \ k = 0, \ C_o = \sqrt{2} (e^{i(\pi/6)}) = \sqrt{2} (\cos \pi/6 + i \sin \pi/6) = \sqrt{2} (\frac{\sqrt{3}}{2} + \frac{i}{2}) = \frac{\sqrt{2}}{2} (\sqrt{3} + i) = \underline{\underline{\frac{\sqrt{6}}{2} + \frac{\sqrt{3}\, i}{2}}}$

$ii) \ If \ \ k = 1, \ C_1 = \sqrt{2} (e^{i(\pi/6 + \pi)}) = \sqrt{2} (\cos 7\pi/6 + i \sin 7\pi/6) = \sqrt{2} (\frac{-\sqrt{3}}{2} - \frac{i}{2}) = \frac{-\sqrt{2}}{2} (\sqrt{3} + i) = \underline{\underline{\frac{-\sqrt{6}}{2} - \frac{\sqrt{3}\, i}{2}}}$

$\therefore$ *The square roots of* $1 + \sqrt{3} i$ *are* $C_o = \frac{\sqrt{6}}{2} + \frac{\sqrt{3}\, i}{2} \quad \& \quad C_1 = \frac{-\sqrt{6}}{2} - \frac{\sqrt{3}\, i}{2}.$

Example 2.61: Find the cube roots of the complex number $z = 8i$.

Solution : *We have* $z = 8i$. *Here* $r = 8,\ \theta = \pi/2,\ n = 3,\ k = 0, 1, 2.$

Hence, $C_k = (r)^{1/n}(e^{i(\frac{\theta}{n}+\frac{2k\pi}{n})})$

$\Rightarrow C_k = (8)^{1/3}(e^{i(\frac{\pi/2}{3}+\frac{2k\pi}{3})})$

$\Rightarrow C_k = 2(e^{i(\pi/6+\frac{2k\pi}{3})})$

$i)$ *If* $k = 0,\ C_o = 2(e^{i(\pi/6)}) = (\cos\pi/6 + i\sin\pi/6) = 2(\frac{\sqrt{3}}{2}+\frac{i}{2}) = \underline{\underline{\sqrt{3}+i}}$

$ii)$ *If* $k = 1,\ C_1 = 2(e^{i(\pi/6+2\pi/3)}) = 2(\cos 5\pi/6 + i\sin 5\pi/6) = 2(\frac{-\sqrt{3}}{2}+\frac{i}{2}) = \underline{\underline{-\sqrt{3}+i}}$

$iii)$ *If* $k = 2,\ C_2 = 2(e^{i(\pi/6+4\pi/3)}) = 2(\cos 3\pi/2 + i\sin 3\pi/2) = 2(0+-i) = \underline{\underline{-2i}}$

$\therefore$ *The cube roots of* $8i$ *are* $C_o = \sqrt{3}+i$, $C_1 = -\sqrt{3}+i$ & $C_2 = -2i$.

Exercise 2.4

1. Find the argument of the following complex numbers:

   $a)\ z = \dfrac{3i}{-1-i}$ $\qquad b)\ z = (\sqrt{3}-i)^6$

2. Show that $a)\ |e^{i\theta}| = 1$ $\qquad b)\ \overline{e^{i\theta}} = e^{-i\theta}$

3. Using mathematical induction, show that $e^{i\theta_1}.\,e^{i\theta_2}.\cdots.e^{i\theta_n} = e^{i(\theta_1+\theta_n+\cdots+\theta_n)}$, $n = 2, 3, \cdots$

4. Show that $a)\ \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ $\qquad b)\ \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

5. Show that $1 + z + z^2 + \cdots + z^n = \dfrac{1-z^{n+1}}{1-z}$, *for* $z \neq 1$.

6. Find the square roots of z = 9i

7. Find the cube roots of z = − 8i

8. Solve the following equations:

   $a)\ z^{3/2} = 8i \quad b)\ z^2 + 4i = 0 \quad c)\ z^2 - 4i = 0$

Chapter 3
Functions

Our everyday lives are filled with situations in which we encounter relationships between two sets. For example,

- To each automobile, there corresponds a license plate number
- To each circle, there corresponds a circumference
- To each number, there corresponds its square

In order to apply mathematics to a variety of disciplines, we must make the idea of a "relationship" between two sets mathematically precise.
On completion of this chapter students will be able to:

- understand the notion of relation and function
- determine the domain and range of relations and functions
- find the inverse of a relation
- define polynomial and rational functions
- perform the fundamental operations on polynomials
- find the inverse of an invertible function
- apply the theorems on polynomials to find the zeros of polynomial functions
- apply theorems on polynomials to solve related problems
- sketch and analyze the graphs of rational functions
- define exponential, logarithmic, trigonometric and hyperbolic functions
- sketch the graph of exponential, logarithmic, trigonometric and hyperbolic functions
- use basic properties of logarithmic, exponential, hyperbolic and trigonometric functions to solve physical problems

In this chapter, we first look at the definitions of relations and functions, and study real valued functions and their properties, types of functions, polynomial functions, zeros of polynomial functions, rational functions and their graphs, logarithmic, exponential, trigonometric and hyperbolic functions and their graphs. Let's begin with the review of relations and functions.

3.1. Review of relations and functions

After completing this section, the student should be able to:

- define Cartesian product of two sets
- understand the notion of relation and function
- know the difference between relation and function
- determine the domain and range of relations and functions
- find the inverse of a relation

The student is familiar with the phrase ordered pair. In the ordered pair $(2,3),(-2,4)$ and (a,b); $2,-2$ and a are the first coordinates while $3,4$ and b are the second coordinates.

- **Cartesian Product**

Given sets $A=\{3,4\}$ and $B=\{2,4,5\}$. Then, the set $\{(3,2),(3,4),(3,5),(4,2),(4,4),(4,5)\}$ is the Cartesian product of A and B, and it is denoted by $A\times B$.

> **Definition 3.1:** Suppose A and B are sets. The Cartesian product of A and B, denoted by $A\times B$, is the set which contains every ordered pair whose first coordinate is an element of A and second coordinate is an element of B, i.e.
> $$A\times B=\{(a,b): a\in A \text{ and } b\in B\}.$$

Example 3.1: For $A=\{2,4\}$ and $A=\{-1,3\}$, we have

a) $A\times B=\{(2,-1),(2,3),(4,-1),(4,3)\}$, and

b) $B\times A=\{(-1,2),(-1,4),(3,2),(3,4)\}$.

Example 3.2: Let $A=\{1,2,3\}$ and $B=\{a,b,c\}$. Then,

$$A\times B=\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\}.$$

From example 3.1, we can see that $A\times B$ and $B\times A$ are not equal. Recall that two sets are equal if one is a subset of the other and vice versa. To check equality of Cartesian products we need to define equality of ordered pairs.

> **Definition 3.2: (Equality of ordered Pairs)**
> Two ordered pairs (a,b) and (c,d) are equal if and only if $a=c$ and $b=d$.

> **Definition 3.3: (Relation from A into B)**
> If A and B are sets, any subset of $A\times B$ is called a relation from A into B.

Suppose R is a relation from a set A to a set B. Then, $R\subseteq A\times B$ and hence for each $(a,b)\in A\times B$, we have either $(a,b)\in R$ or $(a,b)\notin R$. If $(a,b)\in R$, we say "a is R-related (or simply related) to b", and write aRb. If $(a,b)\notin R$, we say that "a is not related to b".
In particular if R is a relation from a set A to itself, then we say that R is a relation on A.

Example 3.3:

1. Let $A=\{1,3,5,7\}$ and $B=\{6,8\}$. Let R be the relation "less than" from A to B. Then, $R=\{(1,6),(1,8),((3,6),(3,8),(5,6),(5,8),(7,8)\}$.
2. Let $A=\{1,2,3,4,5\}$ and $B=\{a,b,c\}$.
   a) The following are relations from A into B;

i) $R_1 = \{(1,a)\}$

ii) $R_2 = \{(2,b),(3,b),(4,c),(5,a)\}$

iii) $R_3 = \{(1,a),((2,b),(3,c)\}$

b) The following are relations from B to A;

i) $R_4 = \{(a,3),(b,1)\}$

ii) $R_5 = \{(b,2),(c,4),(a,2),(b,3)\}$

iii) $R_6 = \{(b,5)\}$

Definition 3.4: Let R be a relation from A into B. Then,

a) the domain of R, denoted by $Dom(R)$, is the set of first coordinates of the elements of R, i.e

$$Dom(R) = \{a \in A : (a,b) \in R\}$$

b) the range of R, denoted by $Range(R)$, is the set of second coordinates of elements of R, i.e

$$Range(R) = \{b \in B : (a,b) \in R\}$$

Remark: If R is a relation from the set A to the set B, then the set B is called the codomain of the relation R. The range of relation is always a subset of the codomain.

Example 3.4:

1. The set $R = \{(4,7),(5,8),(6,10)\}$ is a relation from set $A = \{1,2,3,4,5,6\}$ to set $B = \{6,7,8,9,10\}$. The domain of R is $\{4,5,6\}$, the range of R is $\{7,8,10\}$ and the codomain of R is $\{6,7,8,9,10\}$.
2. The set of ordered pairs $R = \{(8,2),(6,-3),(5,7),(5,-3)\}$ is a relation between the sets $\{5,6,8\}$ and $\{2,-3,7\}$, where $\{5,6,7\}$ is the domain and $\{2,-3,7\}$ is the range.

Remark:

1. If $(a,b) \in R$ for a relation R, we say a is related to (or paired with) b. Note that a may also be paired with an element different from b. In any case, b is called the image of a while a is called the pre-image of b under R.
2. If the domain and/or range of a relation is infinite, we cannot list each element assignment, so instead we use set builder notation to describe the relation. The situation we will encounter most frequently is that of a relation defined by an equation or formula. For example,

$$R = \{(x, y) : y = 2x - 3,\ x, y \in \Re\}$$

is a relation for which the range value is 3 less than twice the domain value. Hence, $(0,-3),(0.5,-2)$ and $(-2,-7)$ are examples of ordered pairs that are of the assignment.

Example 3.5:

1. Let $A = \{1,2,3,4,6\}$

   Let R be the relation on A defined by $R = \{(a,b) : a,b \in A, a \text{ is a factor of } b\}$. Find the domain and range of R.

Solution: We have

$R = \{(1,1),(1,2),(1,3),(1,4),(1,6),(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(6,6)\}$.

Then, $Dom(R) = \{1,2,3,4,6\}$ and $Range(R) = \{1,2,3,4,6\}$.

2. Let $A = \{1,2,3,4,5\}$ and $B = \{1,2,3,\cdots,67\}$.

   Let $R = \{(x,y) \in A \times B : x \text{ is cube root of } y\}$. Find a) R b) $Dom(R)$ c) $Range(R)$

Solution: We have $1 = \sqrt[3]{1},\ 2 = \sqrt[3]{8},\ 3 = \sqrt[3]{27},\ 4 = \sqrt[3]{64},\ 5 = \sqrt[3]{125}$ and $1, 8, 27$ and 64 are in B whereas 125 is not in B. Thus, $R = \{(1,1),(2,8),(3,27),(4,64)\}$, $Dom(R) = \{1,2,3,4\}$ and $R = \{1,8,27,64\}$.

Remark:

1. A relation R on a set A is called
   i) a universal relation if $R = A \times A$
   ii) identity relation if $R = \{(a,a) : a \in A\}$
   iii) void or empty relation if $R = \phi$
2. If R is a relation from A into B, then the inverse relation of R, denoted by R^{-1}, is a relation from B to A and is given by:

$$R^{-1} = \{(y,x) : (x,y) \in R\}.$$

Observe that $Dom(R) = Range(R^{-1})$ and $Range(R) = Dom(R^{-1})$. For instance, if $R = \{(1,4),(9,15),(10,2)\}$ is a relation on a set $A = \{1,2,3,\cdots,20\}$, then $R^{-1} = \{(4,1),(15,9),(2,10)\}$

Example 3.6: Let R be a relation defined on IN by $R = \{(a,b) : a,b \in IN,\ a + 2b = 11\}$.

Find a) R b) $Dom(R)$ c) $Range(R)$ d) R^{-1}

Solution: The smallest natural number is 1.

$b = 1 \quad \Rightarrow \quad a + 2(1) = 11 \quad \Rightarrow a = 9$

$b = 2 \quad \Rightarrow \quad a + 2(2) = 11 \quad \Rightarrow a = 7$

$b = 3 \quad \Rightarrow \quad a + 2(3) = 11 \quad \Rightarrow a = 5$

$b = 4 \quad \Rightarrow \quad a + 2(4) = 11 \quad \Rightarrow a = 3$

$b = 5 \quad \Rightarrow \quad a + 2(5) = 11 \quad \Rightarrow a = 1$

$b = 6 \quad \Rightarrow \quad a + 2(6) = 11 \quad \Rightarrow a = -1 \notin IN$

Therefore, $R = \{(9,1),(7,2),(5,3),(3,4),(1,5)\}$, $Dom(R) = \{1,3,5,7,9\}$, $Range(R) = \{1,2,3,4,5\}$ and $R^{-1} = \{(1,9),(2,7),(3,5),(4,3),(5,1)\}$.

- **Functions**

Mathematically, it is important for us to distinguish among the relations that assign a unique range element to each domain element and those that do not.

Definition 3.5: (Function)
A function is a relation in which each element of the domain corresponds to exactly one element of the range.

Example 3.7: Determine whether the following relations are functions.

a) $R = \{(5,-2),(3,5),(3,7)\}$ b) R $\{(2,4),(3,4),(6,-4)\}$

Solution:

a) Since the domain element 3 is assigned to two different values in the range, 5 and 7, it is not a function.

b) Each element in the domain, $\{2,3,6\}$, is assigned no more than one value in the range, 2 is assigned only 4, 3 is assigned only 4, and 6 is assigned only – 4. Therefore, it is a function.

Remark: Map or mapping, transformation and correspondence are synonyms for the word function. If f is a function and $(x,y)\in f$, we say *x* is mapped to *y* by *f*.

Definition 3.6: A relation f from *A* into *B* is called a function from *A* into *B*, denoted by

$$f : A \to B \quad \text{or} \quad A \xrightarrow{f} B$$

if and only if

(i) $Dom(f) = A$

(ii) No element of *A* is mapped by f to more than one element in *B*, i.e. if $(x,y)\in f$ and $(x,z)\in f$, then $y = z$.

Remark: 1. If to the element *x* of A corresponds $y(\in B)$ under the function f, then we write $f(x) = y$ and y is called the image of *x* under y and *x* is called a pre-image of y under f.

2. The symbol $f(x)$ is read as "f of *x*" but not "f times *x*".

3. In order to show that a relation f from A into B is a function, we first show that the domain of f is *A* and next we show that f well defined or single-valued, i.e. if $x = y$ in *A*, then $f(x) = f(y)$ in *B* for all $x, y \in A$.

Example 3.8:

1. Let $A = \{1,2,3,4\}$ and $B = \{1,6,8,11,15\}$. Which of the following are functions from *A* to B.

   a) f defined by $f(1) = 1, f(2) = 6, f(3) = 8, f(4) = 8$

b) f defined by $f(1)=1, f(2)=6, f(3)=15$

c) f defined by $f(1)=6, f(2)=6, f(3)=6, f(4)=6$

d) f defined by $f(1)=1, f(2)=6, f(2)=8, f(3)=8, f(4)=11$

e) f defined by $f(1)=1, f(2)=8, f(3)=11, f(4)=15$

Solution:

a) f is a function because to each element of A there corresponds exactly one element of B.

b) f is not a function because there is no element of B which correspond to $4(\in A)$.

c) f is a function because to each element of A there corresponds exactly one element of B. In the given function, the images of all element of A are the same.

d) f is not a function because there are two elements of B which correspond to 2. In other words, the image of 2 is not unique.

e) f is a function because to each element of A there corresponds exactly one element of B.

As with relations, we can describe a function with an equation. For example, $y=2x+1$ is a function, since each x will produce only one y.

2. Let $f=\{(x,y): y=x^2\}$. Then, f maps:

1 to 1	-1 to 1
2 to 4	-2 to 4
3 to 9	-3 to 9

More generally any real number x is mapped to its square. As the square of a number is unique, f maps every real number to a unique number. Thus, f is a function from $\Re$ into $\Re$.

We will find it useful to use the following vocabulary: The independent variable refers to the variable representing possible values in the domain, and the dependent variable refers to the variable representing possible values in the range. Thus, in our usual ordered pair notation (x,y), x is the independent variable and y is the dependent variable.

3. Let f be the subset of $Q\times Z$ defined by $f=\left\{\left(\frac{p}{q},p\right): p,q\in Z, q\neq 0\right\}$. Is f a function?

Solution: First we note that $Dom(f)=Q$. Then, f satisfies condition (i) in the definition of a function. Now, $\left(\frac{2}{3},2\right)\in f$, $\left(\frac{4}{6},4\right)\in f$ and $\frac{2}{3}=\frac{4}{6}$ but $f\left(\frac{2}{3}\right)=2\neq 4=f\left(\frac{4}{6}\right)$. Thus f is not well defined. Hence, f is not a function from Q to Z.

4. Let f be the subset of $Z\times Z$ defined by $f=\{(mn,m+n): m,n\in Z\}$. Is f a function?

Solution: First we show that f satisfies condition (i) in the definition. Let x be any element of Z. Then, $x=x\cdot 1$. Hence, $(x,x+1)=(x\cdot 1,x+1)\in f$. This implies that

$x \in Dom(f)$. Thus, $Z \subseteq Dom(f)$. However, $Dom(f) \subseteq Z$ and so $Dom(f) = Z$. Now, $4 \in Z$ and $4 = 4 \cdot 1 = 2 \cdot 2$. Thus, $(4 \cdot 1, 4+1)$ and $(2 \cdot 2, 2+2)$ are in f. Hence we find that $4 \cdot 1 = 2 \cdot 2$ and $f(4 \cdot 1) = 5 \neq 4 = f(2 \cdot 2)$. This implies that f is not well defined, i.e, f does not satisfy condition (ii). Hence, f is not a function from Z to Z.

- **Domain, codomain and range of a function**

For a function $f : A \to B$

(i) The set *A* is called the domain of f

(ii) The set *B* is called the codomain of f

(iii) The set $\{f(x) : x \in A\}$ of all image of elements of *A* is called the range of f

Example 3.9:

1. Let $A = \{1,2,3\}$ and $B = \{1,2,3,\cdots,10\}$. Let $f : A \to B$ be the correspondence which assigns to each element in A, its square. Thus, we have $f(1) = 1, f(2) = 4, f(3) = 9$. Therefore, f is a function and $Dom(f) = \{1,2,3\}$, $Range(f) = \{1,4,9\}$ and codomain of f is $\{1,2,3,\cdots,10\}$.

2. Let $A = \{2,4,6,7,9\}, B = IN$. Let x and y represent the elements in the sets A and B, respectively. Let $f : A \to B$ be a function defined by $f(x) = 15x + 17, x \in A$.

   The variable x can take values 2, 4, 6, 7, 9. Thus, we have
   $$f(2) = 15(2) + 17 = 47, f(4) = 77, f(6) = 107, f(7) = 122, f(9) = 152.$$
   This implies that $Dom(f) = \{2,4,6,7,9\}, Range(f) = \{47,77,107,122,152\}$ and codomain of f is *IN*.

3. Determine whether the following equations determine y as a function of x, if so, find the domain of the function.

   a) $y = -3x + 5$ b) $y = \dfrac{2x}{3x-5}$ c) $y^2 = x$

   Solution:

   a) To determine whether $y = -3x + 5$ gives y as a function of x, we need to know whether each *x-value* uniquely determines a *y-value*. Looking at the equation $y = -3x + 5$, we can see that once x is chosen we multiply it by – 3 and then add 5. Thus, for each *x* there is a unique y. Therefore, $y = -3x + 5$ is a function. It domain is the set of all real numbers.

b) Looking at the equation $y=\dfrac{2x}{3x-5}$ carefully, we can see that each *x-value* uniquely determines a *y-value* (one *x-value* can not produce two different *y-values*). Therefore, $y=\dfrac{2x}{3x-5}$ is a function.

As for its domain, we ask ourselves. Are there any values of x that must be excluded? Since $y=\dfrac{2x}{3x-5}$ is a fractional expression, we must exclude any value of x that makes the denominator equal to zero. We must have

$$3x-5\neq 0 \quad \Leftrightarrow \quad x\neq \frac{5}{3}$$

Therefore, the domain consists of all real numbers except $\frac{5}{3}$. Thus, $Dom(f)=\{x: x\neq \frac{5}{3}\}$.

c) For the equation $y^2=x$, if we choose $x=9$ we get $y^2=9$, which gives $y=\pm 3$. In other words, there are two $y-$values associated with $x=9$. Therefore, $y^2=x$ is not a function.

4. Find the domain of the function $y=\sqrt{3x-x^2}$.

Solution: Since y is defined and is real when the expression under the radical is non-negative, we need x to satisfy the inequality

$$3x-x^2\geq 0 \quad \Leftrightarrow \quad x(3-x)\geq 0$$

This is a quadratic inequality, which can be solved by analyzing signs:

Sign of $3x-x^2$ − − − | + + + | − − −

0   3

Since we want $3x-x^2=x(3-x)$ to be non-negative, the sign analysis shows us that the domain is $\{x: 0\leq x\leq 3\}$ or $[0,3]$.

Exercise 3.1

1. Let *R* be a relation on the set $A=\{1,2,3,4,5,6\}$ defined by $R=\{(a,b): a+b\leq 9\}$.
   - i) List the elements of R
   - ii) Is $R=R^{-1}$
2. Let *R* be a relation on the set $A=\{1,2,3,4,5,6,7\}$ defined by $R=\{(a,b): 4$ divides $a-b\}$.
   - i) List the elements of R
   - ii) Find $Dom(R)$ & $Range(R)$
   - iii) Find the elements of R^{-1}
   - iv) Find $Dom(R^{-1})$ & $Range(R^{-1})$

3. Let $A=\{1,2,3,4,5,6\}$. Define a relation on A by $R=\{(x,y): y=x+1\}$. Write down the domain, codomain and range of R. Find R^{-1}.
4. Find the domain and range of the relation $\{(x,y): |x|+y\geq 2\}$.
5. Let $A=\{1,2,3\}$ and $B=\{3,5,6,8\}$. Which of the following are functions from A to B?
   a) $f=\{(1,3),(2,3),(3,3)\}$
   b) $f=\{(1,3),(2,5),(1,6)\}$
   c) $f=\{(1,8),(2,5)\}$
   d) $f=\{(1,6),(2,5),(3,3)\}$
6. Determine the domain and range of the following relations. Which relation a function?
   a) $\{(-4,-3),(2,-5),(4,6),(2,0)\}$
   b) $\{(8,-2),(6,-\frac{3}{2}),(-1,5)\}$
   c) $\{(-\sqrt{3},3),(-1,1),(0,0),(1,1),(\sqrt{3},3)\}$
   d) $\{(-\frac{1}{2},\frac{1}{6}),(-1,1),(\frac{1}{3},\frac{1}{8})\}$
   e) $\{(0,5),(1,5),(2,5),(3,5),(4,5),(5,5)\}$
   f) $\{(5,0),(5,1),(5,2),(5,3),(5,4),(5,5)\}$
7. Find the domain and range of the following functions.
   a) $f(x)=1+8x-2x^2$
   b) $f(x)=\dfrac{1}{x^2-5x+6}$
   c) $f(x)=\sqrt{x^2-6x+8}$
   d) $f(x)=\begin{cases}3x+4, & -1\leq x<2\\ 1+x, & 2\leq x\leq 5\end{cases}$
8. Given $f(x)=\begin{cases}3x-5, & x<1\\ x^2-1, & x\geq 1\end{cases}$.

   Find a) $f(-3)$ b) $f(1)$ c) $f(6)$

3.2 Real Valued functions and their properties

After completing this section, the student should be able to:

- perform the four fundamental operations on polynomials
- compose functions to get a new function
- determine the domain of the sum, difference, product and quotient of two functions
- define equality of two functions

Let f be a function from set A to set B. If B is a subset of the set of real numbers $\Re$, then f is called a real valued function, and in particular if A is also a subset of $\Re$, then $f: A\rightarrow B$ is called a real function.

Example 3.10: 1. The function $f:\Re\rightarrow\Re$ defined by $f(x)=x^2+3x+7$, $x\in\Re$ is a real function.

2. The function $f:\Re\rightarrow\Re$ defined as $f(x)=|x|$ is also a real valued function.

- **Operations on functions**

Functions are not numbers. But just as two numbers a and b can be added to produce a new number $a+b$, so two functions f and g can be added to produce a new function $f+g$. This is just one of the several operations on functions that we will describe in this section.

Consider functions f and g defined by $f(x)=\frac{x-3}{2}$ and $g(x)=\sqrt{x}$. We can make a new function $f+g$ by having it assign to x the value $\frac{x-3}{2}+\sqrt{x}$, that is,

$$(f+g)(x)=f(x)+g(x)=\frac{x-3}{2}+\sqrt{x}\ .$$

Definition 3.7: Sum, Difference, Product and Quotient of two functions

Let $f(x)$ and $g(x)$ be two functions. We define the following four functions:

1. $(f+g)(x)=f(x)+g(x)$ The sum of the two functions
2. $(f-g)(x)=f(x)-g(x)$ The difference of the two functions
3. $(f\cdot g)(x)=f(x)g(x)$ The product of the two functions
4. $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$ The quotient of the two functions (provided $g(x)\neq 0$)

Since an $x-$value must be an input into both f and g, the domain of $(f+g)(x)$ is the set of all x common to the domain of f and g. This is usually written as $Dom(f+g)=Dom(f)\bigcap Dom(g)$. Similar statements hold for the domains of the difference and product of two functions. In the case of the quotient, we must impose the additional restriction that all elements in the domain of g for which $g(x)=0$ are excluded.

Example 3.11:

1. Let $f(x)=3x^2+2$ and $g(x)=5x-4$. Find each of the following and its domain

a) $(f+g)(x)$ b) $(f-g)(x)$ c) $(f.g)(x)$ d) $\left(\frac{f}{g}\right)(x)$

Solution:

a) $(f+g)(x)=f(x)+g(x)=(3x^2+2)+(5x-4)=3x^2+5x-2$

b) $(f-g)(x)=f(x)-g(x)=(3x^2+2)-(5x-4)=3x^2-5x+6$

c) $(f\cdot g)(x)=(3x^2+2)(5x-4)=15x^3-12x^2+10x-8$

d) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{3x^2+2}{5x-4}$

We have

$$Dom(f+g)=Dom(f-g)=Dom(fg)=Dom(f)\cap Dom(g)=\Re\cap\Re=\Re$$

$$Dom\left(\frac{f}{g}\right)=\left[Dom(f)\cap Dom(g)\right]\setminus\{x:g(x)=0\}=\Re\setminus\left\{\frac{5}{4}\right\}$$

2. Let $f(x)=\sqrt[4]{x+1}$ and $g(x)=\sqrt{9-x^2}$, with respective domains $[-1,\infty)$ and $[-3,3]$. Find formulas for $f+g, f-g, f\cdot g, \frac{f}{g}$ and f^3 and give their domains.

Solution:

Formula	**Domain**
$(f+g)(x)=f(x)+g(x)=\sqrt[4]{x+1}+\sqrt{9-x^2}$	$[-1,3]$
$(f-g)(x)=f(x)-g(x)=\sqrt[4]{x+1}-\sqrt{9-x^2}$	$[-1,3]$
$(f\cdot g)(x)=f(x)\cdot g(x)=\sqrt[4]{x+1}\cdot\sqrt{9-x^2}$	$[-1,3]$
$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{\sqrt[4]{x+1}}{\sqrt{9-x^2}}$	$[-1,3)$
$f^3(x)=(f(x))^3=\left(\sqrt[4]{x+1}\right)^3=(x+1)^{\frac{3}{4}}$	$[-1,\infty)$

There is yet another way of producing a new function from two given functions.

Definition 3.8: (Composition of functions)

Given two functions $f(x)$ and $g(x)$, the composition of the two functions is denoted by $f\circ g$ and is defined by:

$$(f\circ g)(x)=f[g(x)].$$

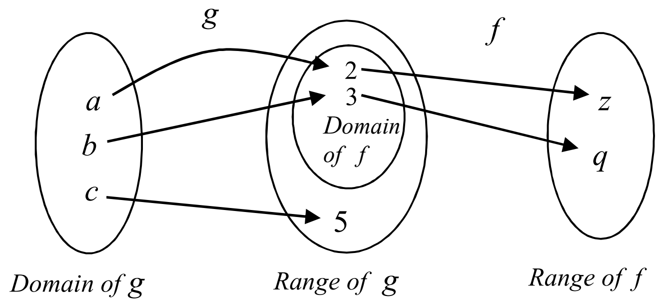
$(f\circ g)(x)$ is read as "f composed with g of x". The domain of $f\circ g$ consists of those x's in the domain of g whose range values are in the domain of f, i.e. those x's for which $g(x)$ is in the domain of f.

Example 3.12:

1. Suppose $f=\{(2,z),(3,q)\}$ and $g=\{(a,2),(b,3),(c,5)\}$. The function $(f\circ g)(x)=f(g(x))$ is found by taking elements in the domain of g and evaluating as follows:

$(f \circ g)(a) = f(g(a)) = f(2) = z, \quad (f \circ g)(b) = f(g(b)) = f(3) = q$

If we attempt to find $f(g(c))$ we get $f(5)$, but 5 is not in the domain of $f(x)$ and so we cannot find $(f \circ g)(c)$. Hence, $f \circ g = \{(a,z),(b,q)\}$. The figure below illustrates this situation.



2. Given $f(x) = 5x^2 - 3x + 2$ and $g(x) = 4x + 3$, find
   a) $(f \circ g)(-2)$ b) $(g \circ f)(2)$ c) $(f \circ g)(x)$ d) $(g \circ f)(x)$

Solution:

a) $(f \circ g)(-2) = f(g(-2))$ …… First evaluate $g(-2) = 4(-2) + 3 = -5$
$= f(-5)$
$= 5(-5)^2 - 3(-5) + 2 = 142$

b) $(g \circ f)(2) = g(f(2))$ …….First evaluate $f(2) = 5(2)^2 - 3(2) + 2 = 16$
$= g(16)$
$= 4(16) + 3 = 67$

c) $(f \circ g)(x) = f(g(x))$ ……. But $g(x) = 4x + 3$
$= f(4x + 3)$
$= 5(4x + 3)^2 - 3(4x + 3) + 2$
$= 80x^2 + 108x + 38$

d) $(g \circ f)(x) = g(f(x))$ ……. But $f(x) = 5x^2 - 3x + 2$
$= g(5x^2 - 3x + 2)$
$= 4(5x^2 - 3x + 2) + 3$
$= 20x^2 - 12x + 11$

3. Given $f(x) = \dfrac{x}{x+1}$ and $g(x) = \dfrac{2}{x-1}$, find
   a) $(f \circ g)(x)$ and its domain b) $(g \circ f)(x)$ and its domain

Solution: a) $(f \circ g)(x) = f\left(\frac{2}{x-1}\right) = \frac{\frac{2}{x-1}}{\frac{2}{x-1}+1} = \frac{2}{x+1}$. Thus, $Dom(f \circ g) = \{x : x \neq \pm 1\}$.

b) $(g \circ f)(x) = g(f(x)) = \frac{2}{\frac{x}{x+1} - 1} = -2x - 2$. Since x must first be an input into $f(x)$ and so must be in the domain of f, we see that $Dom(g \circ f) = \{x : x \neq -1\}$.

4. Let $f(x) = \frac{6x}{x^2 - 9}$ and $g(x) = \sqrt{3x}$. Find $(f \circ g)(12)$ and $(g \circ f)(x)$ and its domain.

Solution: We have $(f \circ g)(12) = f(g(12)) = f(\sqrt{36}) = f(6) = \frac{36}{27} = \frac{4}{3}$.

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{3x}) = \frac{6\sqrt{3x}}{(\sqrt{3x})^2 - 9} = \frac{6\sqrt{3x}}{3x - 9} = \frac{2\sqrt{3x}}{x-3}.$$

The domain of $f \circ g$ is $[0,3) \cup (3, \infty)$.

We now explore the meaning of equality of two functions. Let $f : A \to B$ and $g : A \to B$ be two functions. Then, f and g are subsets of $A \times B$. Suppose $f = g$. Let x be any element of A. Then, $(x, f(x)) \in f = g$ and thus $(x, f(x)) \in g$. Since g is a function and $(x, f(x))$, $(x, g(x)) \in g$, we must have $f(x) = g(x)$. Conversely, assume that $g(x) = f(x)$ for all $x \in A$. Let $(x, y) \in f$. Then, $y = f(x) = g(x)$. Thus, $(x, y) \in g$, which implies that $f \subseteq g$. Similarly, we can show that $g \subseteq f$. It now follows that $f = g$. Thus two functions $f : A \to B$ and $g : A \to B$ are equal if and only if $f(x) = g(x)$ for all $x \in A$. In general we have the following definition.

Definition 3.9: (Equality of functions)
Two functions are said to be equal if and only if the following two conditions hold:
i) The functions have the same domain;
ii) Their functional values are equal at each element of the domain.

Example 3.13:

1. Let $f : Z \to Z^+ \cup \{0\}$ and $g : Z \to Z^+ \cup \{0\}$ be defined by $f = \{(n, n^2) : n \in Z\}$ and $g = \{(n, |n|^2) : n \in Z\}$. Now, for all $n \in Z$, $f(n) = n^2 = |n|^2 = g(n)$. Thus, $f = g$.

2. Let $f(x) = \frac{x^2 - 25}{x - 5}, x \in \Re \setminus \{5\}$, and $g(x) = x + 5, x \in \Re$. The function f and g are not equal because $Dom(f) \neq Dom(g)$.

Exercise 3.2

1. For $f(x) = x^2 + x$ and $g(x) = \frac{2}{x+3}$, find each value:

   a) $(f - g)(2)$ c) $g^2(3)$ e) $(g \circ f)(1)$

   b) $\left(\frac{f}{g}\right)(1)$ d) $(f \circ g)(1)$ f) $(g \circ g)(3)$

2. If $f(x) = x^3 + 2$ and $g(x) = \frac{2}{x-1}$, find a formula for each of the following and state its domain.

   a) $(f + g)(x)$ c) $\left(\frac{g}{f}\right)(x)$

   b) $(f \circ g)(x)$ d) $(g \circ f)(x)$

3. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$.
   a) Find $(f \circ g)(x)$ and its domain.
   b) Find $(g \circ f)(x)$ and its domain
   c) Are $(f \circ g)(x)$ and $(g \circ f)(x)$ the same functions? Explain.
4. Let $f(x) = 5x - 3$. Find $g(x)$ so that $(f \circ g)(x) = 2x + 7$.
5. Let $f(x) = 2x + 1$. Find $g(x)$ so that $(f \circ g)(x) = 3x - 1$.
6. If f is a real function defined by $f(x) = \frac{x-1}{x+1}$. Show that $f(2x) = \frac{3f(x)+1}{f(x)+3}$.
7. Find two functions f and g so that the given function $h(x) = (f \circ g)(x)$, where

   a) $h(x) = (x+3)^3$ c) $h(x) = \frac{1}{x} + 6$

   b) $h(x) = \sqrt{5x - 3}$ d) $h(x) = \frac{1}{x+6}$

8. Let $f(x) = 4x - 3$, $g(x) = \frac{1}{x}$ and $h(x) = x^2 - x$. Find

   a) $f(5x + 7)$ c) $f(g(h(3)))$ e) $f(x + a)$

   b) $5f(x) + 7$ d) $f(1) \cdot g(2) \cdot h(3)$ f) $f(x) + a$

3.3 Types of functions and inverse of a function

After completing this section, the student should be able to:

- define one to oneness and ontoness of a function
- check invertibility of a function
- find the inverse of an invertible function

In this section we shall study some important types of functions.

- **One to One functions**

Definition 3.10: A function $f: A \to B$ is called **one to one**, often written 1 – 1, if and only if for all $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. In words, no two elements of A are mapped to one element of B.

Example 3.14:

1. If we consider the sets $A = \{1,2,3,\cdots,6\}$ and $B = \{7,a,b,c,d,8,e\}$ and if $f = \{(1,7),\ (2,a),\ (3,b),\ (4,b),(5,c),(6,8)\}$ and $g = \{(1,7),(2,a),(3,b),(4,c),(5,8),(6,d)\}$, then both f and g are functions from A into B. Observe that f is not a 1 – 1 function because $f(3) = f(4)$ but $3 \neq 4$. However, g is a 1 – 1 function.

2. Let $A = \{1,2,3,4\}$ and $B = \{1,4,7,8\}$. Consider the functions
   i) $f: A \to B$ defined as $f(1) = 1, f(2) = 4, f(3) = 4, f(4) = 8$
   ii) $g: A \to B$ defined as $f(1) = 4, f(2) = 7, f(3) = 1, f(4) = 8$

   Then, f is not 1 – 1, but g is a 1 – 1 function.

- **Onto functions**

Definition 3.11: Let f be a function from a set A into a set B. Then f is called an **onto function(or f maps onto** B) if every element of B is an image of some element in A, i.e, $Range(f) = B.$

Example 3.15:

1. Let $A = \{1,2,3\}$ and $B = \{1,4,5\}$. The function $f: A \to B$ defined by $f(1) = 1$, $f(2) = 5$, $f(3) = 1$ is not onto because there is no element in A, whose image under f is 4. The function $g: A \to B$ given by $g = \{(1,4),(2,5),(3,1)\}$ is onto because each element of B is an image of at least one element of A.

   Note that if A is a non-empty set, the function $i_A: A \to A$ defined by $i_A(x) = x$ for all $x \in A$ is a 1 – 1 function from A onto A. i_A is called the **identity map** on A.

2. Consider the relation f from Z into Z defined by $f(n) = n^2$ for all $n \in Z$. Now, domain of f is Z. Also, if $n = n'$, then $n^2 = (n')^2$, i.e. $f(n) = f(n')$. Hence, f is well

defined and is a function. However, $f(1)=1=f(-1)$ and $1\neq -1$, which implies that f is not 1 – 1. For all $n\in Z$, $f(n)$ is a non-negative integer. This shows that a negative integer has no preimage. Hence, f is not onto. Note that f is onto $\{0,1,4,9,\cdots\}$.

3. Consider the relation f from Z into Z defined by $f(n)=2n$ for all $n\in Z$. As in the previous example, we can show that f is a function. Let $n,n'\in Z$ and suppose that $f(n)=f(n')$. Then $2n=2n'$ and thus $n=n'$. Hence, f is 1 – 1. Since for all $n\in Z$, $f(n)$ is an even integer; we see that an odd integer has no preimage. Therefore, f is not onto.

- **1 – 1 Correspondence**

> **Definition 3.12:** A function $f:A\rightarrow B$ is said to be a 1 – 1 correspondence if f is both 1 – 1 and onto.

Example 3.16:

1. Let $A=\{0,1,2,3,4,5\}$ and $B=\{0,5,10,15,20,25\}$. Suppose $f:A\rightarrow B$ given by $f(x)=5x$ for all $x\in A$. One can easily see that every element of B has a preimage in A and hence f is onto. Moreover, if $f(x)=f(y)$, then $5x=5y$, i.e. $x=y$. Hence, f is 1 – 1. Therefore, f is a 1 – 1 correspondence between A and B.
2. Let A be a finite set. If $f:A\rightarrow A$ is onto, then it is one to one.

Solution: Let $A=\{a_1,a_2,\cdots,a_n\}$. Then $Range(f)=\{f(a_1),f(a_2),\cdots,f(a_n)\}$. Since f is onto we have $Range(f)=A$.Thus, $A=\{f(a_1),f(a_2),\cdots,f(a_n)\}$, which implies that $f(a_1)$, $f(a_2)$, $\cdots$, $f(a_n)$ are all distinct. Hence, $a_i\neq a_j$ implies $f(a_i)\neq f(a_j)$ for all $1\leq i,j\leq n$. Therefore, f is 1 – 1.

- **Inverse of a function**

Since a function is a relation , the inverse of a function f is denoted by f^{-1} and is defined by:

$$f^{-1}=\{(y,x):(x,y)\in f\}$$

For instance, if $f=\{(2,4),(3,6),(1,7)\}$, then $f^{-1}=\{(4,2),(6,3),(7,1)\}$. Note that the inverse of a function is not always a function. To see this consider the function $f=\{(2,4),(3,6),(5,4)\}$. Then, $f^{-1}=\{(4,2),(6,3),(4,5)\}$, which is not a function.

As we have seen above not all functions have an inverse, so it is important to determine whether or not a function has an inverse before we try to find the inverse. If the function does not have an inverse, then we need to realize that it does not have an inverse so that we do not waste our time trying to find something that does not exist.

A one to one function is special because only one to one functions have inverse. If a function is one to one, to find the inverse we will follow the steps below:

1. Interchange x and y in the equation $y = f(x)$
2. Solving the resulting equation for y, we will obtaining the inverse function.

Note that the domain of the inverse function is the range of the original function and the range of the inverse function is the domain of the original function.

Example 3.17:

1. Given $y = f(x) = x^3$. Find f^{-1} and its domain.

Solution: We begin by interchanging x and y, and we solve for y.

$y = x^3$	Interchange x and y
$x = y^3$	Take the cube root of both sides
$\sqrt[3]{x} = y$	This is the inverse of the function

Thus, $f^{-1}(x) = \sqrt[3]{x}$. The domain of f^{-1} is the set of all real numbers.

2. Let $y = f(x) = \dfrac{x}{x+2}$. Find $f^{-1}(x)$.

Solution: Again we begin by interchanging x and y, and then we solve for y.

$y = \dfrac{x}{x+2}$	Interchange x and y
$x = \dfrac{y}{y+2}$	Solving for y

$$x(y+2) = y \Leftrightarrow xy + 2x = y \Leftrightarrow 2x = y(1-x) \Leftrightarrow y = \frac{2x}{1-x}$$

Thus, $f^{-1}(x) = \dfrac{2x}{1-x}$.

Remark: Even though, in general, we use an exponent of -1 to indicate a reciprocal, inverse function notation is an exception to this rule. Please be aware that $f^{-1}(x)$ is not the reciprocal of f. That is,

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

If we want to write the reciprocal of the function $f(x)$ by using a negative exponent, we must write

$$\frac{1}{f(x)} = [f(x)]^{-1}.$$

Exercise 3.3

1. Consider the function $f = \{(x, x^2) : x \in S\}$ from $S = \{-3,-2,-1,0,1,2,3\}$ into Z. Is f one to one? Is it onto?
2. Let $A = \{1,2,3\}$. List all one to one functions from A onto A.
3. Let $f : A \to B$. Let f^* be the inverse relation, i.e. $f^* = \{(y, x) \in B \times A : f(x) = y\}$.
   a) Show by an example that f^* need not be a function.
   b) Show that f^* is a function from $Range(f)$ into A if and only if f is 1 – 1.
   c) Show that f^* is a function from B into A if and only if f is 1 – 1 and onto.
   d) Show that if f^* is a function from B into A, then $f^{-1} = f^*$.

4. Let $A = \{x \in \Re : 0 \le x \le 1\}$ and $B = \{x \in \Re : 5 \le x \le 8\}$. Show that $f : A \to B$ defined by $f(x) = 5 + (8-5)x$ is a 1 – 1 function from A onto B.

5. Which of the following functions are one to one?
   a) $f : \Re \to \Re$ defined by $f(x) = 4, x \in \Re$
   b) $f : \Re \to \Re$ defined by $f(x) = 6x - 1, x \in \Re$
   c) $f : \Re \to \Re$ defined by $f(x) = x^2 + 7, x \in \Re$
   d) $f : \Re \to \Re$ defined by $f(x) = x^3, x \in \Re$
   e) $f : \Re \setminus \{7\} \to \Re$ defined by $f(x) = \dfrac{2x+1}{x-7}, x \in \Re \setminus \{7\}$
6. Which of the following functions are onto?
   a) $f : \Re \to \Re$ defined by $f(x) = 115x + 49, x \in \Re$
   b) $f : \Re \to \Re$ defined by $f(x) = |x|, x \in \Re$
   c) $f : \Re \to \Re$ defined by $f(x) = \sqrt{x^2}, x \in \Re$
   d) $f : \Re \to \Re$ defined by $f(x) = x^2 + 4, x \in \Re$
7. Find $f^{-1}(x)$ if

   a) $f(x) = 7x - 6$
   b) $f(x) = \dfrac{2x-9}{4}$
   c) $f(x) = 1 - \dfrac{3}{x}$
   d) $f(x) = \dfrac{4-x}{3x}$
   e) $f(x) = \dfrac{5x+3}{1-2x}$
   f) $f(x) = \sqrt[3]{x+1}$
   g) $f(x) == -(x+2)^2 - 1$
   h) $f(x) = \dfrac{2x}{1+x}$

3.4 Polynomials, zeros of polynomials, rational functions and their graphs

After completing this section, the student should be able to:

- define polynomial and rational functions
- apply the theorems on polynomials to find the zeros of polynomial functions
- use the division algorithm to find quotient and remainder
- apply theorems on polynomials to solve related problems
- sketch and analyze the graphs of rational functions

The functions described in this section frequently occur as mathematical models of real-life situations. For instance, in business the demand function gives the price per item, p, in terms of the number of items sold, x. Suppose a company finds that the price p (in Birr) for its model GC-5 calculator is related to the number of calculators sold, x (in millions), and is given by the demand function $p = 80 - x^2$.
The manufacturer's revenue is determined by multiplying the number of items sold (x) by the price per item (p). Thus, the revenue function is

$$R = xp = x(80 - x^2) = 80x - x^3$$

These demand and revenue functions are examples of polynomial functions. The major aim of this section is to better understand the significance of applied functions (such as this demand function). In order to do this, we need to analyze the domain, range, and behavior of such functions.

- **Polynomial functions**

Definition 3.13: A polynomial function is a function of the form

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \ a_n \neq 0.$$

Each a_i is assumed to be a real number, and n is a non-negative integer, a_n is called the leading coefficient. Such a polynomial is said to be of degree n.

Remark:

1. The domain of a polynomial function is always the set of real numbers.
2. (Types of polynomials)
   - A polynomial of degree 1 is called a linear function.
   - A polynomial of degree 2 is called quadratic function.
   - A polynomial of degree 3 is called a cubic function.

     i.e $p(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0, \ a_3 \neq 0.$

Example 3.18: $p(x) = 2x^2 + 1$, $q(x) = \sqrt{3}x^4 + 2x - \pi$ and $f(x) = 2x^3$ are examples of polynomial functions.

- **Properties of polynomial functions**

1. The graph of a polynomial is a smooth unbroken curve. The word smooth means that the graph does not have any sharp corners as turning points.
2. If p is a polynomial of degree n, then it has at most n zeros. Thus, a quadratic polynomial has at most 2 zeros.
3. The graph of a polynomial function of degree n can have at most $n-1$ turning points. Thus, the graph of a polynomial of degree 5 can have at most 4 turning points.
4. The graph of a polynomial always exhibits the characteristic that as $|x|$ gets very large, $|y|$ gets very large.

- **Zeros of a polynomial**

The zeros of a polynomial function provide valuable information that can be helpful in sketching its graph. One can find the zeros by factorizing the polynomial. However, we have no general method for factorizing polynomials of degree greater than 2. In this subsection, we turn our attention to methods that will allow us to find zeros of higher degree polynomials. To do this, we first need to discuss about the division algorithm. Recall that a number a is a zero of a polynomial function p if $p(a)=0$.

Division Algorithm
Let $p(x)$ and $d(x)$ be polynomials with $d(x)\neq 0$, and with the degree of $d(x)$ less than or equal to the degree of $p(x)$. Then there are polynomials $q(x)$ and $R(x)$ such that $\underbrace{p(x)}_{dividend} = \underbrace{d(x)}_{divisor}.\underbrace{q(x)}_{quotient} + \underbrace{R(x)}_{remainder}$, where either $R(x)=0$ or the degree of $R(x)$ is less than degree of $d(x)$.

Example 3.19: Divide $\dfrac{x^4-1}{x^4+2x}$.

Solution: Using long division we have

$$
\begin{array}{r}
x^2-2x+4 \\
x^2+2x\overline{\big)\,x^4+0x^3+0x^2+0x+1} \\
\underline{-(x^4+2x^3)\qquad\qquad\qquad} \\
-2x^3+0x^2\qquad\qquad \\
\underline{-(-2x^3-4x^2)\qquad\qquad} \\
4x^2+0x\qquad \\
\underline{-(4x^2+8x)\qquad} \\
-8x-1
\end{array}
$$

This long division means $\underbrace{x^4-1}_{dividend}=\underbrace{(x^2+2x)}_{divisor}.\underbrace{(x^2-2x+4)}_{quotient}+\underbrace{(-8x-1)}_{remainder}$.

With the aid of the division algorithm, we can derive two important theorems that will allow us to recognize the zeros of polynomials.

If we apply the division algorithm where the divisor, $d(x)$, is linear (that is of the form $x-r$), we get

$$p(x)=(x-r)q(x)+R$$

Note that since the divisor is of the first degree, the remainder R, must be a constant. If we now substitute $x=r$, into this equation, we get

$$P(r)=(r-r)q(r)+R=0\cdot q(r)+R$$

Therefore, $p(r)=R$.

The result we just proved is called the remainder theorem.

The Remainder Theorem
When a polynomial $p(x)$ of degree at least 1 is divided by $x-r$, then the remainder is $p(r)$.

Example 3.20: The remainder when $P(x)=x^3-x^2+3x-1$ is divided by $x-2$ is $p(2)=9$.

As a consequence of the remainder theorem, if $x-r$ is a factor of $p(x)$, then the remainder must be 0. Conversely, if the remainder is 0, then $x-r$, is a factor of $p(x)$. This is known as the Factor Theorem.

The Factor Theorem
$x-r$ is a factor of $p(x)$ if and only if $p(r)=0$.

The next theorem, called location theorem, allows us to verify that a zero exists somewhere within an interval of numbers, and can also be used to zoom in closer on a value.

Location theorem
Let f be a polynomial function and a and b be real numbers such that $a < b$. If $f(a)f(b) < 0$, then there is at least one zero of f between a and b.

The Factor and Remainder theorems establish the intimate relationship between the factors of a polynomial $p(x)$ and its zeros. Recall that a polynomial of degree *n* can have at most *n* zeros. Does every polynomial have a zero? Our answer depends on the number system in which we are working. If we restrict ourselves to the set of real number system, then we are already familiar with the fact that the polynomial $p(x) = x^2 + 1$ has no real zeros. However, this polynomial does have two zeros in the complex number system. (The zeros are i and $-i$). Carl Friedrich Gauss (1777-1855), in his doctoral dissertation, proved that within the complex number system, every polynomial of degree ≥ 1 has at least one zero. This fact is usually referred to as the Fundamental theorem of Algebra.

Fundamental Theorem of Algebra
If $p(x)$ is a polynomial of degree $n > 0$ whose coefficients are complex numbers, then $p(x)$ has at least one zero in the complex number system.

Note that since all real numbers are complex numbers, a polynomial with real coefficients also satisfies the Fundamental theorem of Algebra. As an immediate consequence of the Fundamental theorem of Algebra, we have

The linear Factorization Theorem
If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $n \geq 1$ and $a_n \neq 0$, then $p(x) = a_n(x - r_1)\,(x - r_2)\cdots(x - r_n)$, where the r_i are complex numbers (possible real and not necessarily distinct).

From the linear factorization theorem, it follows that every polynomial of degree $n \geq 1$ has exactly n zeros in the complex number system, where a root of multiplicity k counted k times.

Example 3.21: Express each of the polynomials in the form described by the Linear Factorization Theorem. List each zero and its multiplicity.

a) $p(x) = x^3 - 6x^2 - 16x$

b) $q(x) = 3x^2 - 10x + 8$

c) $f(x) = 2x^4 + 8x^3 + 10x^2$

Solution:

a) We may factorize $p(x)$ as follows:

$$\begin{aligned} p(x) = x^3 - 6x^2 - 16x &= x(x^2 - 6x - 16) \\ &= x(x-8)(x+2) \\ &= x(x-8)(x-(-2)) \end{aligned}$$

The zeros of $p(x)$ are 0, 8, and – 2 each of multiplicity one.

b) We may factorize $q(x)$ as follows:

$$q(x) = 3x^2 - 10x + 8 = (3x-4)\ (x-2)$$
$$= 3(x - \frac{4}{3})(x-2)$$

Thus, the zeros of $q(x)$ are $\frac{4}{3}$ and 2, each of multiplicity one.

c) We may factorize $f(x)$ as follows:

$$f(x) = 2x^4 + 8x^3 + 10x^2 = 2x^2(x^2 + 4x + 5)$$
$$= 2x^2(x-(-2+i))(x-(-2-i))$$

Thus, the zeros of f(x) are 0 with multiplicity two and $-2+i$ and $-2-i$ each with multiplicity one.

Example 3.22:

1. Find a polynomial $p(x)$ with exactly the following zeros and multiplicity.

zeros	multiplicity
-1	3
2	4
5	2

Are there any other polynomials that give the same roots and multiplicity?

2. Find a polynomial $f(x)$ having the zeros described in part (a) such that $f(1) = 32$.

Solution:

1. Based on the Factor Theorem, we may write the polynomial as:
$$p(x) = (x-(-1))^3\,(x-2)^4(x-5)^2 = (x+1)^3\,(x-2)^4(x-5)^2$$
which gives the required roots and multiplicities.
Any polynomial of the form $kp(x)$, where k is a non-zero constant will give the same roots and multiplicities.
2. Based on part (1), we know that $f(x) = k(x+1)^3\,(x-2)^4(x-5)^2$. Since we want $f(x) = 32$, we have
$$f(1) = k(1+1)^3\,(1-2)^4(1-5)^2$$
$$32 = k(8)(1)(16) \quad \Rightarrow k = \tfrac{1}{4}$$
Thus, $f(x) = \tfrac{1}{4}(x+1)^3\,(x-2)^4(x-5)^2$.

Our experience in using the quadratic formula on quadratic equations with real coefficients has shown us that complex roots always appear in conjugate pairs. For example, the roots of $x^2 - 2x + 5 = 0$ are $1+2i$ and $1-2i$. In fact, this property extends to all polynomial equations with real coefficients.

Conjugate Roots Theorem
Let $p(x)$ be a polynomial with real coefficients. If complex number $a+bi$ (where a and b are real numbers) is a zero of $p(x)$, then so is its conjugate $a-bi$.

Example 3.23: Let $r(x)=x^4+2x^3-9x^2+26x-20$. Given that $1-\sqrt{3}\ i$ is a zero, find the other zero of $r(x)$.

Solution: According to the Conjugate Roots Theorem, if $1-\sqrt{3}\ i$ is a zero, then its conjugate, $1+\sqrt{3}\ i$ must also be a zero. Therefore, $x-(1-\sqrt{3}\ i)$ and $x-(1+\sqrt{3}\ i)$ are both factors of $r(x)$, and so their product must be a factor of $r(x)$. That is, $[x-(1-\sqrt{3}\ i)]\,[x-(1+\sqrt{3}\ i)]=$ x^2-2x+4 is a factor of $r(x)$. Dividing $r(x)$ by x^2-2x+4, we obtain

$$r(x)=(x^2-2x+4)(x^2+4x-5)=(x^2-2x+4)\,(x+5)\,(x-1).$$

Thus, the zeros of $r(x)$ are $1-\sqrt{3}\,i$, $1+\sqrt{3}\,i$, -5 and 1.

The theorems we have discussed so far are called existence theorems because they ensure the existence of zeros and linear factors of polynomials. These theorems do not tell us how to find the zeros or the linear factors. The Linear Factorization Theorem guarantees that we can factor a polynomial of degree at least one into linear factors, but it does not tell us how.

We know from experience that if $p(x)$ happens to be a quadratic function, then we may find the zeros of $p(x)=Ax^2+Bx+C$ by using the quadratic formula to obtain the zeros

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

The rest of this subsection is devoted to developing some special methods for finding the zeros of a polynomial function.

As we have seen, even though we have no general techniques for factorizing polynomials of degree greater than 2, if we happen to know a root, say r, we can use long division to divide $p(x)$ by $x-r$ and obtain a quotient polynomial of lower degree. If we can get the quotient polynomial down to a quadratic, then we are able to determine all the roots. But how do we find a root to start the process? The following theorem can be most helpful.

The Rational Root Theorem
Suppose that $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $n\geq 1$, $a_n\neq 0$ is an n^{th} degree polynomial with integer coefficients. If $\frac{p}{q}$ is a rational root of $f(x)=0$, where p and q have no common factor other than ± 1, then p is a factor of a_0 and q is a factor of a_n.

To get a feeling as to why this theorem is true, suppose $\frac{3}{2}$ is a root of $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$.

Then, $a_3\left(\frac{3}{2}\right)^3 + a_2\left(\frac{3}{2}\right)^2 + a_1\left(\frac{3}{2}\right) + a_0 = 0$ which implies that

$$\frac{27a_3}{8} + \frac{9a_2}{4} + \frac{3a_1}{2} + a_0 = 0 \qquad \text{multiplying both sides by 8}$$

$$27a_3 + 18a_2 + 12a_1 = -8a_0 \text{..(1)}$$

$$27a_3 = -18a_2 - 12a_1 - 8a_0 \text{..(2)}$$

If we look at equation (1), the left hand side is divisible by 3, and therefore the right hand side must also be divisible by 3. Since 8 is not divisible by 3, a_0 must be divisible by 3. From equation (2), a_3 must be divisible by 2.

Example 3.24: Find all the zeros of the function $p(x) = 2x^3 + 3x^2 - 23x - 12$.

Solution: According to the Rational Root Theorem, if $\frac{p}{q}$ is a rational root of the given equation, then p must be a factor of -12 and q must be a factor of 2. Thus, we have

possible values of p: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

possible values of q: $\pm 1, \pm 2$

possible rational roots $\frac{p}{q}$: $\pm 1, \pm\frac{1}{2}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm 4, \pm 6, \pm 12$

We may check these possible roots by substituting the value in $p(x)$. Now $p(1) = -30$ and $p(-1) = 12$. Since $p(1)$ is negative and $p(-1)$ is positive, by location theorem, $p(x)$ has a zero between -1 and 1. Since $P\left(-\frac{1}{2}\right) = 0$, then $\left(x + \frac{1}{2}\right)$ is a factor of $p(x)$. Using long division, we obtain

$$p(x) = 2x^3 + 3x^2 - 23x - 12 = (x + \tfrac{1}{2})(2x^2 + 2x - 24)$$
$$= 2(x + \tfrac{1}{2})(x + 4)(x - 3)$$

Therefore, the zeros of $p(x)$ are $-\frac{1}{2}$, -4 and 3.

- **Rational Functions and their Graphs**

A rational function is a function of the form $f(x) = \frac{n(x)}{d(x)}$ where both $n(x)$ and $d(x)$ are polynomials and $d(x) \neq 0$.

Example 3.25: The functions $f(x)=\frac{3}{x+5}$, $f(x)=\frac{x-1}{x^2-4}$ and $f(x)=\frac{x^5+2x^3-x+1}{x \ +5x}$ are examples of rational function.

Note that the domain of the rational function $f(x)=\frac{n(x)}{d(x)}$ is $\{x: d(x)\neq 0\}$

Example 3.26: Find the domain and zeros of the function $f(x)=\frac{3x-5}{x^2-x-12}$.

Solution: The values of x for which $x^2-x-12=0$ are excluded from the domain of f. Since $x^2-x-12=(x-4)(x+3)$, we have $Dom(f)=\{x: x\neq -3,4\}$. To find the zeros of $f(x)$, we solve the equation

$$\frac{n(x)}{d(x)}=0 \Leftrightarrow n(x)=0 \ \& \ q(x)\neq 0$$

Therefore, to find the zeros of $f(x)$, we solve $3x-5=0$, giving $x=\frac{5}{3}$. Since $\frac{5}{3}$ does not make the denominator zero, it is the only zero of $f(x)$.

The following terms and notations are useful in our next discussion.

Given a number a,

- x approaches a from the right means x takes any value near and near to a but $x > a$. This is denoted by: $x \to a^+$ (read: 'x approaches a from the right').

  For instance, $x \to 1^+$ means x can be 1.001, 1.0001, 1.00001, 1.000001, etc.

- x approaches a from the left means x takes any value near and near to a but $x < a$.

  This is denoted by: $x \to a^-$ (read: 'x approaches a from the left').

  For instance, $x \to 1^-$ means x can be 0.99, 0.999, 0.9999, 0.9999, etc.

- $x\to\infty$ (read: 'x approaches or tends to *infinity*') means the value of x gets indefinitely larger and larger in magnitude (keep increasing without bound). For instance, x can be 10^6, 10^{10}, 10^{12}, etc.

- $x\to -\infty$ (read: 'x approaches or tends to negative *infinity*') means the value of x is negative and gets indefinitely larger and larger negative in magnitude (keep decreasing without bound). For instance, x can be -10^6, -10^{10}, -10^{12}, etc.

The same meanings apply also for the values of a function f if we wrote $f(x)\to\infty$ or $f(x)\to-\infty$. The following figure illustrates these notion and notations.

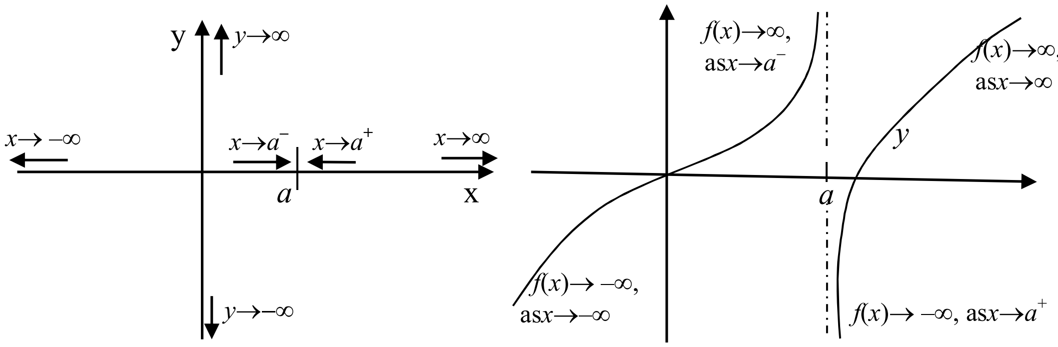


Fig. 2.1. Graphical illustration of the idea of $x\to a^+$, $f(x)\to\infty$, etc.

We may also write $f(x)\to b$ (read: '$f(x)$ approaches b') to mean the function values, $f(x)$, becomes arbitrarily closer and closer to b (i.e., approximately b) but not exactly equal to b. For instance, if $f(x)=\frac{1}{x}$, then $f(x)\to 0$ as $x\to\infty$; i.e., $\frac{1}{x}$ is approximately 0 when x is arbitrarily large.

The following steps are usually used to sketch (or draw) the graph of a rational function $f(x)$.

1. Identify the domain and simplify it.
2. Find the intercepts of the graph whenever possible. Recall the following:
   - y–intercept is the point on y-axis where the graph of $y = f(x)$ intersects with the y-axis. At this point x=0. Thus, $y = f(0)$, or $(0, f(0))$ is the y-intercept if $0\in\text{Dom}(f)$.
   - x–intercept is the point on x-axis where the graph of $y = f(x)$ intersects with the x-axis. At this point y=0. Thus, $x=a$ or $(a, 0)$ is x-intercept if $f(a)=0$.
3. Determine the asymptotes of the graph. Here, remember the following.

- Vertical Asymptote: The vertical line $x=a$ is called a vertical asymptote(VA) of $f(x)$ if
  i) $a\notin\text{dom}(f)$, i.e., f is not defined at $x=a$; and
  ii) $f(x)\to\infty$ or $f(x)\to -\infty$ when $x\to a^+$ or $x\to a^-$. In this case, the graph of f is almost vertically rising upward (if $f(x)\to\infty$) or sinking downward (if $f(x)\to-\infty$) along with the vertical line $x=a$ when x approaches a either from the right or from the left.

Example 3.27: Consider $f(x)=\frac{1}{(x-a)^n}$, where $a \geq 0$ and n is a positive integer.

Obviously $a\notin\text{Dom}(f)$. Next, we investigate the trend of the values of $f(x)$ near a. To do this, we consider two cases, when n is even or odd:

Suppose n is even: In this case $(x-a)^n > 0$ for all $x\in\Re\setminus\{a\}$; and since $(x-a)^n\to 0$ as $x\to a^+$ or $x\to a^-$. Hence, $f(x)=\frac{1}{(x-a)^n}\to\infty$ as $x\to a^+$ or $x\to a^-$. Therefore, $x=a$ is a VA of $f(x)$. Moreover, $y= 1/a^n$ or $(0, 1/a^n)$ is its y-intercept since $f(0)=1/a^n$. However, it has no x-intercept since $f(x) >0$ for all x in its domain (See, Fig. 2.2 (A)).

Suppose n is odd: In this case $(x-a)^n > 0$ for all $x>a$ and $1/(x-a)^n \to \infty$ when $x \to a^+$ as in the above case. Thus, $x=a$ is its VA. However, $1/(x-a)^n \to -\infty$ when $x \to a^-$ since $(x-a)^n < 0$ for $x<a$. Moreover, $y=-1/a^n$ or $(0, -1/a^n)$ is its y-intercept since $f(0) = -1/a^n$. However, it has no x-intercept also in this case. (See, Fig. 2.2 (B)).

Note that in both cases, $f(x) = \frac{1}{(x-a)^n} \to 0$ as $x \to \infty$ or $x \to -\infty$.

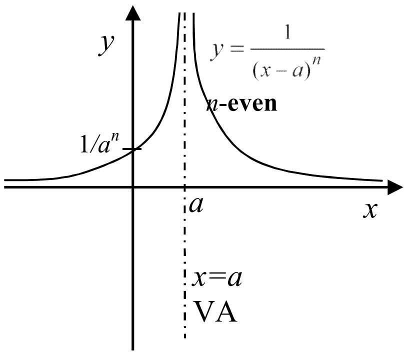


Fig. 2.2 (A)

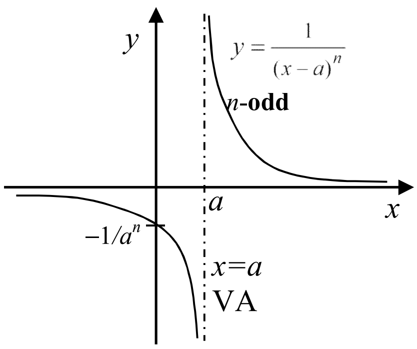


Fig. 2.2 (B)

Remark: Let $f(x) = \frac{n(x)}{d(x)}$ be a rational function. Then,

1. if $d(a) = 0$ and $n(a) \neq 0$, then $x=a$ is a VA of f.
2. if $d(a) = 0 = n(a)$, then $x=a$ may or may **not** be a VA of f. In this case, simplify $f(x)$ and look for VA of the simplest form of f.

- Horizontal Asymptote: A horizontal line $y=b$ is called horizontal asymptote (HA) of $f(x)$ if the value of the function becomes closer and closer to b (i.e., $f(x) \to b$) as $x \to \infty$ or as $x \to -\infty$.

  In this case, the graph of f becomes almost a horizontal line along with (or near) the line $y=b$ as $x \to \infty$ and as $x \to -\infty$. For instance, from the above example, the HA of $f(x) = \frac{1}{(x-a)^n}$ is $y=0$ (the x-axis), for any positive integer n (See, Fig. 2.2).

Remark: A rational function $f(x) = \frac{n(x)}{d(x)}$ has a HA only when *degree*($n(x)$) ≤ *degree*($d(x)$).

In this case, (i) If *degree*($n(x)$) < *degree*($d(x)$), then $y = 0$ (the x-axis) is the HA of f.

(ii) If *degree*($n(x)$) = *degree*($d(x)$) = n, i.e., $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$,

then $y = \frac{a_n}{b_n}$ is the HA of f.

- Oblique Asymptote: The oblique line $y=ax+b$, $a \neq 0$, is called an oblique asymptote (OA) of f if the value of the function, $f(x)$, becomes closer and closer to $ax+b$ (i.e., $f(x)$ becomes approximately $ax+b$) as either $x \to \infty$ or $x \to -\infty$. In this case, the graph of f becomes almost a straight line along with (or near) the oblique line $y=ax+b$ as $x \to \infty$ and as $x \to -\infty$.

Note: A rational function $f(x)=\frac{n(x)}{d(x)}$ has an OA only when *degree*(*n*(*x*)) = *degree*(*d*(*x*)) + 1. In this case, using long division, if the quotient of $n(x) \div d(x)$ is $ax+b$, then $y=ax+b$ is the OA of *f*.

Example 3.28: Sketch the graphs of (a) $f(x)=\frac{x+2}{x-1}$ (b) $g(x)=\frac{x^2+3x+2}{x^2-1}$

Solution: (a) Since $x-1=0$ at $x=1$, dom(*f*) = $\Re\backslash\{1\}$.

- Intercepts: y-intercept: $x=0 \Rightarrow y=f(0)=-2$. Hence, (0, – 2) is y-intercept.
  x-intercept: $y=0 \Rightarrow x+2=0 \Rightarrow x=-2$. Hence, (–2, 0) is x-intercept.
- Asymptotes:
  - VA: Since $x-1=0$ at $x=1$ and $x+2\neq 0$ at $x=1$, $x=1$ is VA of *f*. In fact, if $x\to 1^+$, then $x+2\approx 3$ but the denominator $x-1$ is almost 0 (but positive).
    Consequently, $f(x)\to\infty$ as $x\to 1^+$.
    Moreover, $f(x)\to -\infty$ as $x\to 1^-$ (since , if $x\to 1^-$ then $x-1$ is almost 0 but negative) .

    (So, the graph of *f* rises up to $+\infty$ at the right side of $x=1$, and sink down to $-\infty$ at the left side of $x=1$)
  - HA: Note that if you divide $x+2$ by $x-1$, the quotient is 1 and remainder is 3. Thus, $f(x)=\frac{x+2}{x-1}=1+\frac{3}{x-1}$. Thus, if $x\to\infty$ (or $x\to -\infty$), then $\frac{3}{x-1}\to 0$ so that $f(x)\to 1$. Hence, $y=1$ is the HA of *f*.

  Using these information, you can sketch the graph of *f* as displayed below in Fig. 2.3 (A).

(b) Both the denominator and numerator are 0 at $x=1$. So, first factorize and simplify them: $x^2+3x+2=(x+2)(x+1)$ and $x^2-1=(x-1)(x+1)$. Therefore,

$$g(x)=\frac{x^2+3x+2}{x^2-1}=\frac{(x+2)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}},\quad x\neq -1$$

$$=\frac{x+2}{x-1}.\qquad \text{(So, dom}(g)=\Re\backslash\{1,-1\}\text{)}$$

This implies that only $x=1$ is VA.

Hence, the graph of $g(x)=\frac{x+2}{x-1}$, $x\neq -1$, is exactly the same as that of $f(x)=\frac{x+2}{x-1}$ except that $g(x)$ is not defined at $x=-1$. Therefore, the graph of *g* and its VA are the same as that of *f* except that there should be a 'hole' at the point corresponding to $x=-1$ on the graph of *g* as shown on Fig. 2.3(B) below.

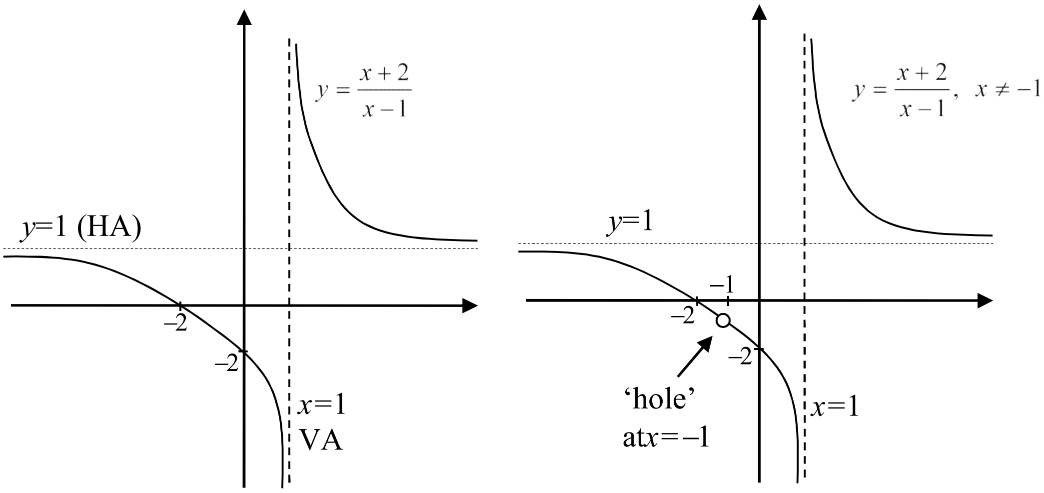


Fig 2.3 (A) $f(x) = \frac{x+2}{x-1}$ (B)

Exercise 3.4

1. Perform the requested divisions. Find the quotient and remainder and verify the Remainder Theorem by computing $p(a)$.
   a) Divide $p(x) = x^2 - 5x + 8$ *by* $x + 4$
   b) Divide $p(x) = 2x^3 - 7x^2 + x + 4$ *by* $x - 4$
   c) Divide $p(x) = 1 - x^4$ *by* $x - 1$
   d) Divide $p(x) = x^5 - 2x^2 - 3$ *by* $x + 1$
2. Given that $p(4) = 0$, factor $p(x) = 2x^3 - 11x^2 + 10x + 8$ as completely as possible.
3. Given that $r(x) = 4x^3 - x^2 - 36x + 9$ and $r\left(\frac{1}{4}\right) = 0$, find the remaining zeros of $r(x)$.
4. Given that 3 is a double zero of $p(x) = x^4 - 3x^3 - 19x^2 + 87x - 90$, find all the zeros of $p(x)$.
5. a) Write the general polynomial $p(x)$ whose only zeros are 1, 2 and 3, with multiplicity 3, 2 and 1 respectively. What is its degree?
   b) Find $p(x)$ described in part (a) if $p(0) = 6$.
6. If $2 - 3i$ is a root of $p(x) = 2x^3 - 5x^2 + 14x + 39$, find the remaining zeros of p(x).
7. Determine the rational zeros of the polynomials
   a) $p(x) = x^3 - 4x^2 - 7x + 10$
   b) $p(x) = 2x^3 - 5x^2 - 28x + 15$
   c) $p(x) = 6x^3 + x^2 - 4x + 1$
8. Find the domain and the real zeros of the given function.

a) $f(x)=\dfrac{3}{x^2-25}$ b) $g(x)=\dfrac{x-3}{x^2 4x-12}$ c) $f(x)=\dfrac{(x-3)^2}{x^3-3x^2+2x}$ d) $f(x)=\dfrac{x^2-16}{x^2+4}$

9. Sketch the graph of

a) $f(x)=\dfrac{1-x}{x-3}$ b) $f(x)=\dfrac{x^2+1}{x}$ c) $f(x)=\dfrac{1}{x}+2$ d) $f(x)=\dfrac{x^2}{x^2-4}$

10. Determine the behavior of $f(x)=\dfrac{x^3-8x-3}{x-3}$ when x is near 3.

11. The graph of any rational function in which the degree of the numerator is exactly one more than the degree of the denominator will have an oblique (or slant) asymptote.
    a) Use long division to show that

    $$y=f(x)=\frac{x^2-x+6}{x-2}=x+1+\frac{8}{x-2}$$

    b) Show that this means that the line $y=x+1$ is a slant asymptote for the graph and sketch the graph of $y=f(x)$.

3.5 Definition and basic properties of logarithmic, exponential, trigonometric and hyperbolic functions and their graphs

After completing this section, the student should be able to:

- define exponential, logarithmic, trigonometric and hyperbolic functions
- understand the relationship of the exponential and logarithmic functions
- define the hyperbolic functions and be familiar with their properties
- sketch the graph of exponential, logarithmic, trigonometric and hyperbolic functions
- use basic properties of logarithmic, exponential, hyperbolic and trigonometric functions to solve problems

- **Exponents and radicals**

Definition 3.14: For a natural number n and a real number x, the power x^n, read " the n^{th} power of x" or "x raised to n", is defined as follows:

$$x^n = \underbrace{x\cdot x\cdot\cdots\cdot x}_{n\ \text{factors each equal to}\ x}$$

In the symbol x^n, x is called the base and n is called the exponent.

For example, $2^5=2\times2\times2\times2\times2=32$.

Based of the definition of x^n, n must be a natural number. It does not make sense for n to be negative or zero. However, we can extend the definition of exponents to include 0 and negative exponents.

Definition 3.15: (Zero and Negative Exponents)

Definition of zero Exponent

$$x^0 = 1 \ (x \neq 0)$$

Definition of Negative Exponent

$$x^{-n} = \frac{1}{x^n} \quad (x \neq 0)$$

Note: 0^0 is undefined.

As a result of the above definition, we have $\frac{1}{x^{-n}} = x^n$. We have the following rules of exponents for integer exponents:

Rules for Integer Exponents

1. $x^n \cdot x^m = x^{n+m}$
2. $(x^n)^m = x^{nm}$
3. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \quad (y \neq 0)$
4. $(xy)^n = x^n y^n$
5. $\frac{x^n}{x^m} = x^{n-m}$

Next we extend the definition of exponents even further to include rational number exponents. To do this, we assume that we want the rules for integer exponents also to apply to rational exponents and then use the rules to show us to define a rational exponent. For example, how do we define $a^{\frac{1}{2}}$? Consider $9^{\frac{1}{2}}$.

If we apply rule 2 and square $9^{\frac{1}{2}}$, we get $\left(9^{\frac{1}{2}}\right)^2 = 9^{\frac{2}{2}} = 9$. Thus, $9^{\frac{1}{2}}$ is a number that, when squared, yields 9. There are two possible answers: 3 and – 3, since squaring either number will yield 9. To avoid ambiguity, we define $a^{\frac{1}{2}}$ (called the principal square root of a) as the non-negative quantity that, when squared, yield a. Thus, $9^{\frac{1}{2}} = 3$.

We will arrive at the definition of $a^{\frac{1}{3}}$ in the same way as we did for $a^{\frac{1}{2}}$. For example, if we cube $8^{\frac{1}{3}}$, we get $\left(8^{\frac{1}{3}}\right)^3 = 8^{\frac{3}{3}} = 8$. Thus, $8^{\frac{1}{3}}$ is the number that, when cubed, yields 8. Since $2^3 = 8$ we have $8^{\frac{1}{3}} = 2$. Similarly, $(-27)^{\frac{1}{3}} = -3$. Thus, we define $a^{\frac{1}{3}}$ (called the cube root of a) as the quantity that, when cubed yields a.

Definition 3.16: (Rational Exponent $a^{\frac{1}{n}}$)

If n is an odd positive integer, then $a^{\frac{1}{n}} = b$ if and only if $b^n = a$

If n is an even positive integer and $a \geq 0$, then $a^{\frac{1}{n}} = |b|$ if and only if $b^n = a$

We call $a^{\frac{1}{n}}$ the principal n^{th} root of a. Hence, $a^{\frac{1}{n}}$ is the real number (nonnegative when n is even) that, when raised to the n^{th} power, yields a. Therefore,

$$(16)^{\frac{1}{2}} = 4 \text{ since } 4^2 = 16$$

$$(-125)^{\frac{1}{3}} = -5 \text{ since } (-5)^3 = -125$$

$$\left(\frac{1}{81}\right)^{\frac{1}{4}} = \frac{1}{3} \text{ since } \left(\frac{1}{3}\right)^4 = \frac{1}{81}$$

$$27^{\frac{1}{3}} = 3 \text{ since } 3^3 = 27$$

$$(-16)^{\frac{1}{4}} \text{ is not a real number}$$

Thus far, we have defined $a^{\frac{1}{n}}$, where n is a natural number. With the help of the second rule for exponent, we can define the expression $a^{\frac{m}{n}}$, where m and n are natural numbers and $\frac{m}{n}$ is reduced to lowest terms.

Definition 3.17: (Rational Exponent $a^{\frac{m}{n}}$)

If $a^{\frac{1}{n}}$ is a real number, then $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$ (i.e. the n^{th} root of a raised to the m^{th} power)

We can also define negative rational exponents:

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} \quad (a \neq 0)$$

Example 3.29: Evaluate the following

a) $27^{\frac{2}{3}}$ b) $36^{-\frac{1}{2}}$ c) $(-32)^{-\frac{3}{5}}$

Solution: We have

a) $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = 3^2 = 9$

b) $36^{-\frac{1}{2}} = \frac{1}{36^{\frac{1}{2}}} = \frac{1}{6}$

c) $(-32)^{-\frac{3}{5}} = \dfrac{1}{(-32)^{\frac{3}{5}}} = \dfrac{1}{\left((-32)^{\frac{1}{5}}\right)^3} = \dfrac{1}{(-2)^3} = -\dfrac{1}{8}$

Radical notation is an alternative way of writing an expression with rational exponents. We define for real number a, the n^{th} root of a as follows:

Definition 3.18 (n^{th} root of a): $\sqrt[n]{a} = a^{\frac{1}{n}}$, where n is a positive integer.

The number $\sqrt[n]{a}$ is also called the principal n^{th} root of a. If the n^{th} root of a exists, we have:

For a a real number and n a positive integer,

$$\sqrt[n]{a^n} = \begin{cases} |a|, & \text{if } n \text{ is even} \\ a, & \text{if } n \text{ is odd} \end{cases}$$

For example, $\sqrt[3]{5^3} = 5$ and $\sqrt[4]{(-3)^4} = 3$.

- **Exponential Functions**

In the previous sections we examined functions of the form $f(x) = x^n$, where n is a constant. How is this function different from $f(x) = n^x$.

Definition 3.19: A function of the form $y = f(x) = b^x$, where $b > 0$ and $b \neq 1$, is called an exponential function.

Example 3.30: The functions $f(x) = 2^x$, $g(x) = 3^x$ and $h(x) = \left(\dfrac{1}{2}\right)^x$ are examples of exponential functions.

As usual the first question raised when we encounter a new function is its domain. Since rational exponents are well defined, we know that any rational number will be in the domain of an exponential function. For example, let $f(x) = 3^x$. Then as x takes on the rational values $x = 4$, -2, $\frac{1}{2}$ and $\frac{4}{5}$, we have

$$f(4) = 3^4 = 3\cdot 3\cdot 3\cdot 3 = 81 \qquad f(-2) = 3^{-2} = \tfrac{1}{3^2} = \tfrac{1}{9}$$

$$f(\tfrac{1}{2}) = 3^{\frac{1}{2}} = \sqrt{3} \qquad f(\tfrac{4}{5}) = 3^{\frac{4}{5}} = \sqrt[5]{3^4} = \sqrt[5]{81}$$

Note that even though we do not know the exact values of $\sqrt{3}$ and $\sqrt[5]{81}$, we do know exactly what they mean. However, what about $f(x)$ for irrational values of x? For instance, $f(\sqrt{2}) = 3^{\sqrt{2}} = ?$

We have not defined the meaning of irrational exponents. In fact, a precise formal definition of b^x where x is irrational requires the ideas of calculus. However, we can get an idea of what $3^{\sqrt{2}}$ should be by using successive rational approximations to $\sqrt{2}$. For example, we have

$$1.414 < \sqrt{2} < 1.415$$

Thus, it would seem reasonable to expect that $3^{1.414} < 3^{\sqrt{2}} < 3^{1.415}$. Since 1.414 and 1.415 are rational numbers, $3^{1.414}$ and $3^{1.415}$ are well defined, even though we cannot compute their values by hand. Using a calculator, we get $4.7276950 < 3^{\sqrt{2}} < 4.7328918$. If we use better approximations to $\sqrt{2}$, we get $3^{1.4142} < 3^{\sqrt{2}} < 3^{1.4143}$. Using a calculator again, we get $4.7287339 < 3^{\sqrt{2}} < 4.7292535$. Computing $3^{\sqrt{2}}$ directly on a calculator gives $3^{\sqrt{2}} \approx 4.7288044$. This numerical evidence suggests that as x approaches $\sqrt{2}$, the values of 3^x approach a unique real number that we designate by $3^{\sqrt{2}}$, and so we will accept without proof, the fact that the domain of the exponential function is the set of real numbers.

The exponential function $y = b^x$, where $b > 0$ and $b \neq 1$, is defined for all real values of x. In addition all the rules for rational exponents hold for real number exponents as well.

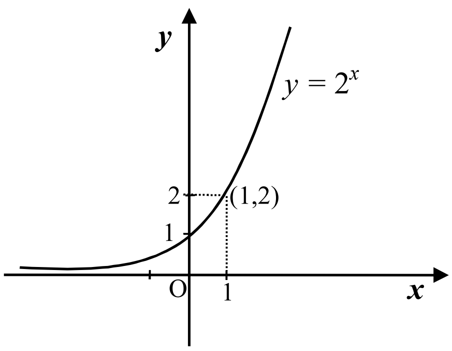
Before we state some general facts about exponential functions , let's see if we can determine what the graph of an exponential function will look like.

Example 3.31:

1. Sketch the graph of the function $y = 2^x$ and identify its domain and range.

   Solution: To aid in our analysis, we set up a short table of values to give us a frame of reference.

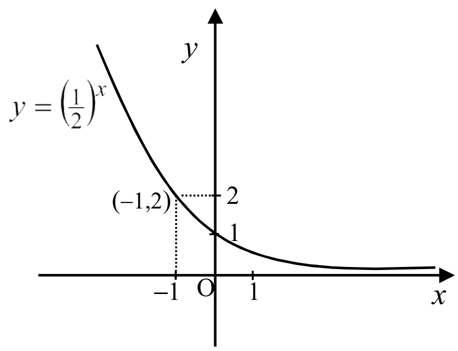
x	y
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



With these points in hand, we draw a smooth curve through the points obtaining the graph appearing above. Observe that the domain of $y = 2^x$ is IR, the graph has no $x-$intercepts, as $x \to +\infty$, the y values are increasing very rapidly, whereas as $x \to -\infty$, the y values are getting closer and closer to 0. Thus, $x-axis$ is a horizontal asymptote, the $y-$intercept is 1 and the range of $y = 2^x$ is the set of positive real numbers.

2. Sketch the graph of $y = f(x) = \left(\frac{1}{2}\right)^x$.

Solution: It would be instructive to compute a table of values as we did in example 1 above (you are urged to do so). However, we will take a different approach. We note that $y = f(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$. If $f(x) = 2^x$, then $f(-x) = 2^{-x}$. Thus by the graphing principle for $f(-x)$, we can obtain the graph of $y = 2^{-x}$ by reflecting the graph of $y = 2^x$ about the $y-axis$.



Here again the $x-axis$ is a horizontal asymptote, there is no $x-$intercept, 1 is $y-$intercept and the range is the set of positive real numbers. However, the graph is now decreasing rather than increasing.

The following box summarizes the important facts about exponential functions and their graphs.

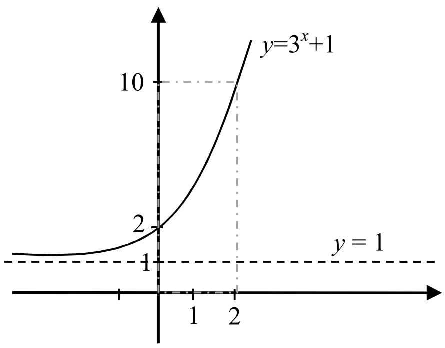
The Exponential function $y = f(x) = b^x$
1. The domain of the exponential function is the set of real numbers 2. The range of the exponential function is the set of positive real numbers 3. The graph of $y = b^x$ exhibits exponential growth if $b > 1$ or exponential decay if $0 < b < 1$. 4. The $y-$intercept is 1. 5. The $x-$intercept is a horizontal asymptote 6. The exponential function is 1 – 1. Algebraically if $b^x = b^y$, then $x = y$

Example 3.32: Sketch the graph of each of the following. Find the domain, range, intercepts, and asymptotes.

a) $y = 3^x + 1$ b) $y = 3^{x+1}$ c) $y = -9^{-x} + 3$

Solution:

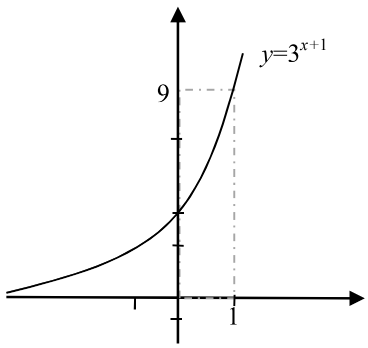
a) To get the graph of $y = 3^x + 1$. We start with the graph of $y = 3^x$, which is the basic exponential growth graph, and shift it up 1 unit.



From the graph we see that

- $Dom(f) = \Re$
- $Range(f) = (1, \infty)$
- The $y-$intercept is 2
- The line $y = 1$ is a horizontal asymptote

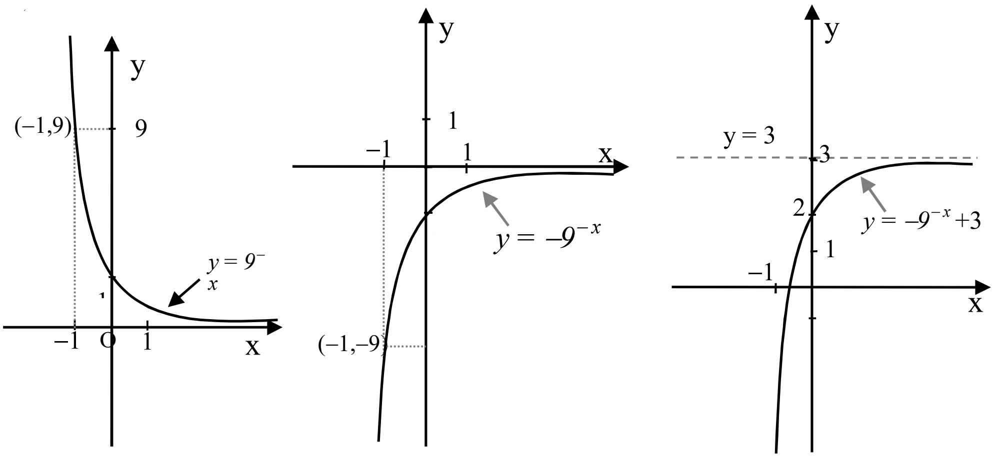
b) To get the graph of $y = 3^{x+1}$, we start with the graph of $y = 3^x$, and shift 1 unit to the left.



From the graph we see that

- $Dom(f) = \Re$
- $Range(f) = (0, \infty)$
- The $y-$intercept is 3
- The line $y = 0$ is a horizontal asymptote

c) To get the graph of $y = -9^{-x} + 3$, we start with the basic exponential decay $y = 9^{-x}$. We then reflect it with respect to the $x-axis$, which gives the graph of $y = -9^{-x}$. Finally, we shift this graph up 3 units to get the required graph of $y = -9^{-x} + 3$.



From the graph of $y=-9^{-x}+3$, we can see that $Dom(h)=\Re$, $Range(h)=(-\infty,3)$, the line $y=3$ is a horizontal asymptote, 2 is the $y-$intercept and $x=-\frac{1}{2}$ is the $x-$intercept.

Remark: When the base b of the exponential function $f(x)=b^x$ equals to the number e, where $e=2.7182\cdots$, we call the exponential function the natural exponential function.

- **Logarithmic Functions**

In the previous subsection we noted that the exponential function $f(x)=b^x$ (where $b>0$ and $b\neq 1$) is one to one. Thus, the exponential function has an inverse function. What is the inverse of $f(x)=b^x$?

To find the inverse of $f(x)=b^x$, let's review the process for finding an inverse function by comparing the process for the polynomial function $y=x^3$ and the exponential function $y=3^x$. Keep in mind that x is our independent variable and y is the dependent variable and so whenever possible we want a function solved explicitly for y.

To find the inverse of $y=x^3$		**To find the inverse of** $y=3^x$	
$y=x^3$	Interchange x and y	$y=3^x$	Interchange x and y
$x=y^3$	solve for y	$x=3^y$	solve for y
$y=\sqrt[3]{x}$		$y=??$	

There is no algebraic procedure we can use to solve $x=3^y$ for y. By introducing radical notations we could express the inverse of $y=x^3$ explicitly in the form $y=\sqrt[3]{x}$. In words, $y^3=x$ and $y=\sqrt[3]{x}$ both mean exactly the same thing: y is the number whose cube is x. Similarly, if we want to express $x=3^y$ explicitly as a function of x, we need to invent a special notation for this. The key idea is to take the equation $x=3^y$ and express it verbally.

$x = 3^y$ means y is the exponent to which 3 must be raised to yield x

We introduce the following notation, which expresses this idea in a much more compact form.

Definition 3.20: For $b > 0$ and $b \neq 1$, we write $y = \log_b x$ to mean y is the exponent to which b must be raised to yield x. In other words,

$$x = b^y \Leftrightarrow y = \log_b x$$

We read $y = \log_b x$ as "y equals the logarithm of x to the base b".

REMEMBER: $y = \log_b x$ is an alternative way of writing $x = b^y$

When an expression is written in the form $x = b^y$, it is said to be in exponential form. When an expression is written in the form $y = \log_b x$, it is said to be in logarithmic form. The table below illustrates the equivalence of the exponential and logarithmic forms.

Exponential form	Logarithmic form
$4^2 = 16$	$\log_4 16 = 2$
$2^4 = 16$	$\log_2 16 = 4$
$5^{-3} = \frac{1}{125}$	$\log_5 \frac{1}{125} = -3$
$6^{\frac{1}{2}} = \sqrt{6}$	$\log_6 \sqrt{6} = \frac{1}{2}$
$7^0 = 1$	$\log_7 1 = 0$

Example 3.33:

1. Write each of the following in exponential form.

   a) $\log_3^{\frac{1}{9}} = -2$ b) $\log_{16} 2 = \frac{1}{4}$

   Solution: We have a) $\log_3^{\frac{1}{9}} = -2$ means $3^{-2} = \frac{1}{9}$ and b) $\log_{16} 2 = \frac{1}{4}$ means $16^{\frac{1}{4}} = 2$

2. Write each of the following in logarithmic form.

   a) $10^{-3} = 0.001$ b) $27^{\frac{2}{3}} = 9$

   Solution: We have a) $10^{-3} = 0.001$ means $\log_{10} 0.001 = -3$

   b) $27^{\frac{2}{3}} = 9$ means $\log_{27} 9 = \frac{2}{3}$

3. Evaluate each of the following.

   a) $\log_3 81$ b) $\log_8 \frac{1}{64}$

Solution:

a) To evaluate $\log_3 81$, we let $t = \log_3 81$, and then rewrite the equation in exponential form, $3^t = 81$. Now, if we can express both sides in terms of the same base, we can solve the resulting exponential equation, as follows:

Let	$t = \log_3 81$	Rewrite in exponential form
	$3^t = 81$	Express both sides in terms of the same base
	$3^t = 3^4$	Since the exponential function is 1 – 1
	$t = 4$	

Therefore, $\log_3 81 = 4$.

b) We apply the same procedure as in part (a).

Let	$t = \log_8 \frac{1}{64}$	Rewrite in exponential form
	$8^t = \frac{1}{64}$	Express both sides in terms of the same base
	$8^t = 8^{-2}$	Since the exponential function is 1 – 1
	$t = -2$	

Therefore, $\log_8 \frac{1}{64} = -2$.

As was pointed out at the beginning of this subsection, logarithm notation was invented to express the inverse of the exponential function. Thus, $\log_b x$ is a function of x. We usually write $f(x) = \log_b x$ rather than writing $f(x) = \log_b (x)$ and use parenthesis only when needed to clarify the input to the log function. For example,

If $f(x) = \log_5 (4 - x)$, then $f(-1) = \log_5 (4-(-1)) = \log_5 5 = 1$, whereas if $f(x) = 4 - \log_5 x$, then $f(-1) = 4 - \log_5 (-1)$, which is undefined.

Example 3.34: Given $f(x) = \log_5 x$, find

a) $f(25)$ b) $f(\frac{1}{25})$ c) $f(0)$ d) $f(-125)$

Solution:

a) $f(25) = \log_5 25 = 2$ (since $5^2 = 25$)

b) $f(\frac{1}{25}) = \log_5 \frac{1}{25} = -2$ (since $5^{-2} = \frac{1}{25}$)

c) $f(0) = \log_5 0$ is not defined (what power of 5 will yield 0?). We say that 0 is not in the domain of f.

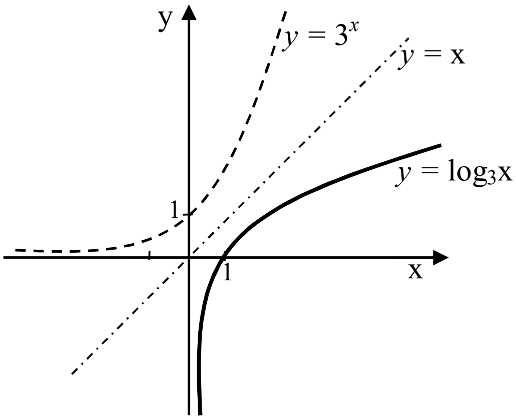
d) $f(-125) = \log_5 (-125)$ is not defined (what power of 5 will yield -125?). We say that -125 is not in the domain of f.

Acknowledging that the logarithmic and exponential functions are inverses, we can derive a great deal of information about the logarithmic function and its graph from the exponential function and its graph.

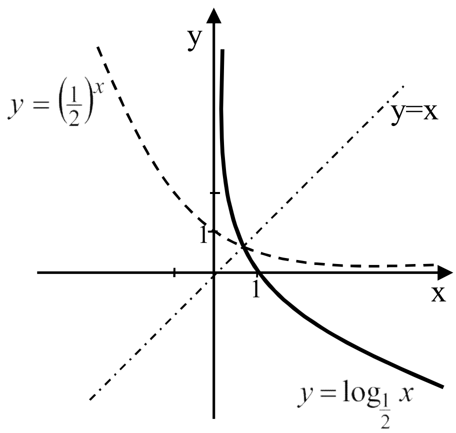
Example 3.35: Sketch the graph of the following functions. Find the domain and range of each.

a) $y = \log_3 x$ b) $y = \log_{\frac{1}{2}} x$

Solution: a) Since $y = \log_3 x$ is the inverse of $y = 3^x$, we can obtain the graph of $y = \log_3 x$ by reflecting the graph of $y = 3^x$ about the line $y = x$, as shown below.



b) To get the graph of $y = \log_{\frac{1}{2}} x$, we reflect the graph of $y = \left(\frac{1}{2}\right)^x$ about the line $y = x$ as shown below.



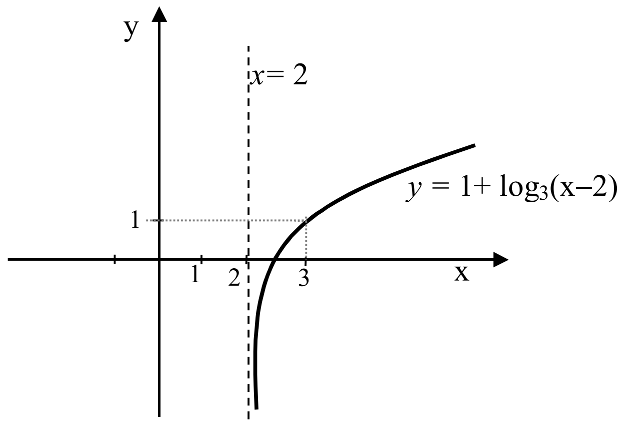
Taking note of the features of the two graphs we have the following important informations about the graph of the log function:

The Logarithmic Function $y = \log_b x$
1. Its domain is the set of positive real numbers 2. Its range is the set of real numbers. 3. Its graph exhibits logarithmic growth if $b > 1$ and logarithmic decay if $0 < b < 1$. 4. The $x-$ intercept is 1. There is no $y-$intercept. 5. The $y-axis$ is a vertical asymptote.

Example 3.36:

1. Sketch the graph of $f(x)=1+\log_3(x-2)$. Find the domain, range, asymptote and intercepts.

   Solution: We can obtain the graph of $y=1+\log_3(x-2)$ by applying the graphing principle to shift the basic logarithmic growth graph 2 units to the right and 1 unit up.



   We have $Dom(f)=\{x:x>2\}$, $Range(f)=\Re$ and the graph has the line $x=2$ as a vertical asymptote. To find the intercept, we set $y=0$ and solve for x. Setting $y=0$ and solving for x, we will obtain $x=\frac{7}{3}$. Thus, the $x-$intercept is $\frac{7}{3}$.

2. Find the inverse function for

   a) $y=f(x)=3^x+4$ b) $y=g(x)=\log_3(x-2)$

   Solution: Following the procedure for finding an inverse function, we have

(a) $y=3^x+4$ Interchange x and y

$x=3^y+4$ solve explicitly for y

$x-4=3^y$ Write in logarithmic form

$y=\log_3(x-4)$

Thus, $f^{-1}(x)=\log_3(x-4)$

(b) $y=\log_3(x-2)$ Interchange x and y

$x=\log_3(y-2)$ Write in logarithmic form

$y-2=3^x$ solve explicitly for y

$y=3^x+2$

Thus, $g^{-1}(x)=3^x+2$

The following table contains the basic properties of logarithm:

Properties of logarithm

Assume that b,u and v are positive and $b\neq 1$. Then

1. $\log_b(uv)=\log_b u+\log_b v$

   In words, logarithm of a product is equal to the sum of the logs of the factors.

2. $\log_b(\frac{u}{v})=\log_b u-\log_b v$

   In words, the log of a quotient is the log of the numerator minus the log of the denominator.

3. $\log_b\left(u^r\right) = r\log_b u$

   In words, the log of a power is the exponent times the log.

4. $\log_b(b^x) = x\log_b b = x$

5. $b^{\log_b x} = x$

6. $\log_b c = \dfrac{\log_a c}{\log_a b}$ if a is positive and $a \neq 1$.

Example 3.37:

1. Express in terms of simpler logarithms.

   a) $\log_b(x^3 y)$ b) $\log_b(x^3 + y)$ c) $\log_b\left(\frac{\sqrt{xy}}{z^3}\right)$

   Solution:

   a) $\log_b(x^3 y) = \log_b x^3 + \log_b y = 3\log_b x + \log_b y$

   b) Examining the properties of logarithms, we can see that they deal with log of a product, quotient and power. Thus, $\log_3(x^3 + y)$ which is the log of a sum cannot be simplified using log properties.

   c) We have

   $$\log_b\left(\frac{\sqrt{xy}}{z^3}\right) = \log_b \sqrt{xy} - \log_b(z^3) = \log_b(xy)^{\frac{1}{2}} - 3\log_b z = \frac{1}{2}(\log_b x + \log_b y) - 3\log_b z\,.$$

2. Show that $\log_b \frac{1}{2} = -\log_b 2$.

   Solution: We have $\log_b \frac{1}{2} = \log_b 1 - \log_b 2 = 0 - \log_b 2 = -\log_b 2$.

The logarithmic function was introduced without stressing the particular base chosen. However, there are two bases of special importance in science and mathematics, namely, $b = 10$ and $b = e$.

Definition 3.21: (Common Logarithm)
$f(x) = \log_{10} x$ is called the common logarithm function. We write $\log_{10} x = \log x$.

The inverse of the natural exponential function is called the natural logarithmic function and has its own special notation.

Definition 3.22: (Natural Logarithm)
$f(x) = \log_e x$ is called the natural logarithmic function. We write $\log_e x = \ln x$.

Example 3.38:

1. Evaluate $\log 1000$

Solution: Let $a = \log 1000$. Then, $a = \log_{10} 1000 = \log_{10}(10^3) = 3$.

2. Find the inverse function of $f(x) = e^x + 1$.

Solution: Let $y = e^x + 1$ Interchange x and y

$x = e^y + 1$ Solve for y

$x - 1 = e^y$ Rewrite in logarithmic form

$y = \ln(x-1)$

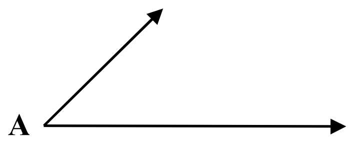
Thus, $f^{-1}(x) = \ln(x-1)$.

- **Trigonometric functions and their graphs**

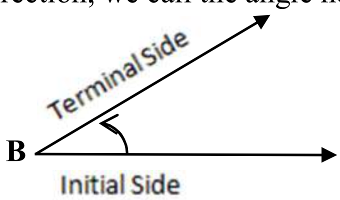
For the functions we have encountered so far, namely polynomial, rational and exponential functions, as the independent variable goes to infinity the graph of each of these three functions either goes to infinity(very quickly) for exponential functions or approaches a finite horizontal asymptote. None of these functions can model the regular periodic patterns that play an important role in the social, biological, and physical sciences: business cycles, agricultural seasons, heart rhythms, and hormone level fluctuations, and tides and planetary motions. The basic functions for studying regular periodic behaviour are the trigonometric functions. The domain of the trigonometric functions is more naturally the set of all geometric angles.

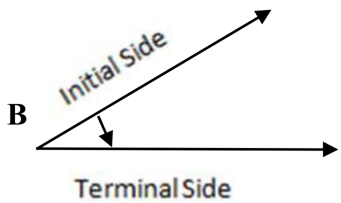
Angle Measurement

An angle is the figure formed by two half-lines or rays with a common end point. The common end point is called the vertex of the angle.



In forming the angle, one side remains fixed and the other side rotates. The fixed side is called the initial side and the side that rotates is called the terminal side. If the terminal side rotates in a counter clockwise direction, we call the angle positive angle, and if the terminal side rotates in a clockwise direction, we call the angle negative angle.

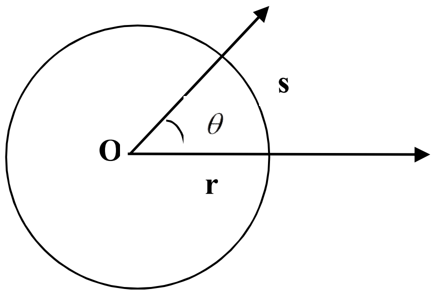




What attribute of an angle are we trying to measure when we measure the size of an angle? A moment of thought will lead us to the conclusion that when we measure an angle we are trying to answer the question: Through what part of a complete rotation has the terminal side rotated?

We will use degree (°) as the unit of measurement for angles. Recall that the measure of a full round angle (full circle) is 360°, straight angle is 180°, and right angle is 90°.

An alternative unit of measure for angles which will indicate their size is the radian measure. To see the connection between the degree measure and radian measure of an angle, let us consider an angle θ and draw a circle of radius r with the vertex of θ at its center O. Let s represent the length of the arc of the circle intercepted by $\angle\theta$ (as shown below).



Basic geometry tells us that the central angle θ will be the same fractional part of one complete rotation as s will be of the circumference of the circle. For example, if θ is $\frac{1}{10}$ of a complete rotation, then s will be $\frac{1}{10}$ of the circumference of the circle. In other words, we can set up the following proportion:

$$\frac{\theta}{1\;complete\;rotation} = \frac{s}{circumference\;of\;circle} = \frac{s}{2\pi r}$$

Thus, we have the following conversion formula:

$$\frac{\theta\;in\;degrees}{180^\circ} = \frac{\theta\;in\;radians}{\pi}$$

Example 3.39:

1. Convert each of the following radian measures to degrees.

   a) $\frac{\pi}{6}$ b) $\frac{3\pi}{5}$

Solution: a) By the conversion formula, we have $\frac{\theta}{180^\circ} = \frac{\frac{\pi}{6}}{\pi}$, which implies that $\theta = 30^\circ$.

b) Again using the conversion formula, we get $\frac{\theta}{180^\circ} = \frac{\frac{3\pi}{5}}{\pi}$, which implies that $\theta = 108^\circ$.

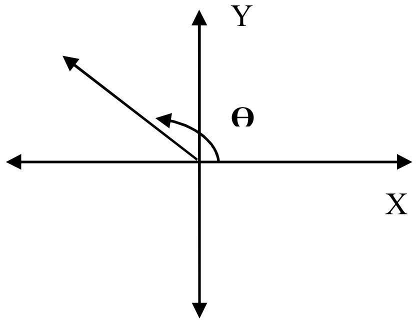
2. Convert to radian measures

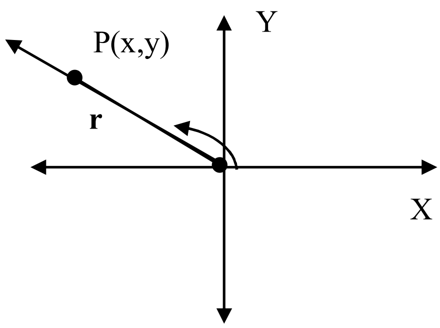
   a) 90° b) 270°

Solution: a) Let θ represent the radian measure of 90°. Using the conversion formula, we obtain: $\frac{\theta}{\pi} = \frac{90^\circ}{180^\circ}$, which implies that $\theta = \frac{\pi}{2}$.

b) Rather than using the conversion formula, we notice that $270^\circ = 3(90^\circ)$. In part (a) we found that $90^\circ = \frac{\pi}{2}$, and so we have $270^\circ = \frac{3\pi}{2}$.

To define the trigonometric functions, we will view all angles in the context of a Cartesian coordinate system: that is, given an angle θ, we begin by putting θ in standard position, meaning that the vertex of θ is placed at the origin and initial side of θ is placed along the positive $x-axis$. Thus the location of the terminal side of θ will, of course, depend on the size of θ.





We then locate a point(other than the origin) on the terminal side of θ and identify its coordinates (x, y) and its distance to the origin, dented by r. Then, r is positive.
With θ in standard position, we define the six trigonometric functions of θ as follows:

Definition 3.23

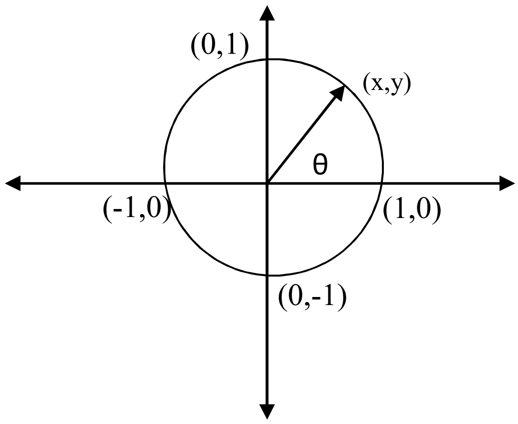
Name of function	**Abbreviation**	**Definition**
Sine θ	$\sin\theta$	$\sin\theta = \frac{y}{r}$
Cosine θ	$\cos\theta$	$\cos\theta = \frac{x}{r}$
Tangent θ	$\tan\theta$	$\tan\theta = \frac{y}{x}$
Cosecant θ	$\csc\theta$	$\csc\theta = \frac{r}{y}$
Secant θ	$\sec\theta$	$\sec\theta = \frac{r}{x}$
Cotangent θ	$\cot\theta$	$\cot\theta = \frac{x}{y}$

Recall that the radian measure of an angle is defined as $\theta = \frac{s}{r}$, where θ is angle in radians s is the length of the arc intercepted by θ and r is the length of the radius. Since s and r are both lengths, the quotient $\frac{s}{r}$ is a pure number without any units attached. Thus, any angle can be interpreted as a real number. Conversely, any real number can be interpreted as an angle. Thus, we can describe the domains of the trigonometric functions in the frame work of the real number systems. If we let $f(\theta)=\sin\theta$, the domain consists of all real numbers θ for which $\sin\theta$ is defined. Since $\sin\theta=\frac{y}{r}$ and r is never equal to zero, the domain for $\sin\theta$ is the set of all real numbers. Similarly, the domain of $f(\theta)=\cos\theta=\frac{x}{r}$ is also the set of all real numbers.

- **The graph of** $y=\sin\theta$

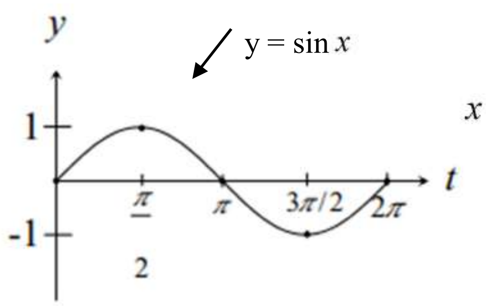
To analyze $f(\theta)=\sin\theta$, we keep in mind that once we choose a real number θ , we draw the angle, in standard position, that corresponds to θ . To simplify our analysis, we choose the point (x,y) on the terminal side so that $r=1$. That is, (x,y) is a point on the unit circle $x^2+y^2=1$.

Note that $\sin\theta=\frac{y}{1}=y$.

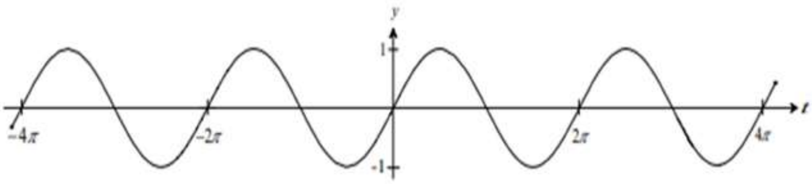


As the terminal side of θ moves through the first quadrant, y increases from 0 (when $\theta=0$) to 1(when $\theta=\frac{\pi}{2}$). Thus, as θ increases from 0 to $\frac{\pi}{2}$, $y=\sin\theta$ steadily increases from 0 to 1. As θ increases from $\frac{\pi}{2}$ to π , $y=\sin\theta$ decreases form 1 to 0. A similar analysis reveals that as θ increases from π to $\frac{3\pi}{2}$, $\sin\theta$ decreases from 0 to – 1; and as θ increases from $\frac{3\pi}{2}$ to 2π , $\sin\theta$ increases from – 1 to 0.

Based on this analysis, we have the graph of $f(\theta)=\sin\theta$ in the interval $[0,2\pi]$ as show below.



Since the values of $f(\theta)=\sin\theta$ depend only on the position of the terminal side, adding or subtracting multiples of 2π to θ will leave the value of $f(\theta)=\sin\theta$ unchanged. Thus, the values of $f(\theta) = \sin\theta$ will repeat every 2π units. The complete graph of $f(\theta) = \sin\theta$ appears below.



The graph of $y = \sin x$, which is called the basic sine curve.

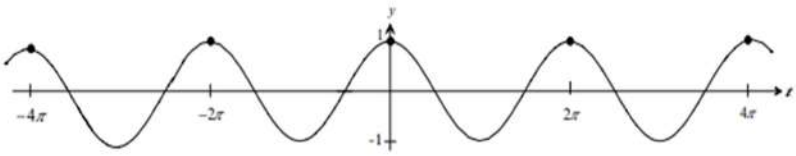
- **The graph of** $y = \cos\theta$

Applying the same type of analysis to $f(\theta) = \cos\theta$, we will able to get a good idea of what its graph looks like. The figure below shows the angle corresponding to θ as it increases through quadrant I, II, III and IV.

Keeping in mind that $\cos\theta = \frac{x}{1} = x$, we have the following:

1. As θ increases from 0 to $\frac{\pi}{2}$, $x = \cos\theta$ decreases from 1 to 0.
2. As θ increases from $\frac{\pi}{2}$ to π, $x = \cos\theta$ decreases from 0 to – 1.
3. As θ increases from π to $\frac{3\pi}{2}$, $x = \cos\theta$ increases from – 1 to 0.
4. As θ increases from $\frac{3\pi}{2}$ to 2π , $x = \cos\theta$ increases from 0 to 1.

Based on this analysis, we have the graph of $f(\theta) = \cos\theta$ as shown below:

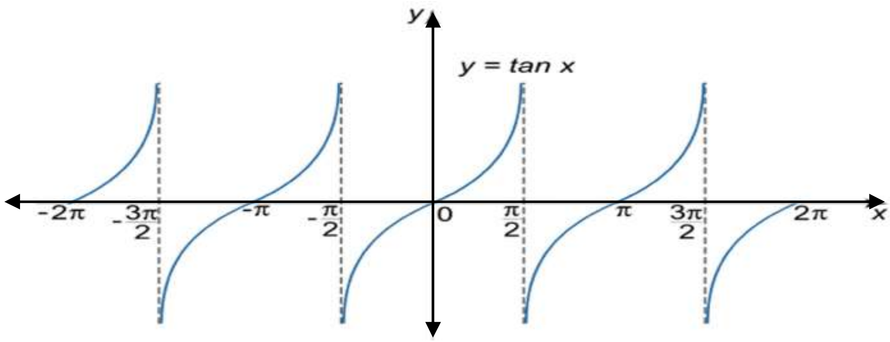


- **The graph of** $y = \tan\theta$

Since $\tan\theta = \frac{y}{x}$ is undefined whenever $x = 0$, $\tan\theta$ is undefined whenever the terminal side of the angle corresponding to θ falls on the $y-axis$. This happens for $\theta = \frac{\pi}{2}$, to which we can add or subtract any multiple of π that will again bring the terminal side back to the $y-axis$. Thus, domain of $\tan\theta$ is $\{\theta : \theta \neq \frac{\pi}{2} + n\pi\}$, where n is an integer.

1. As θ increases from 0 to $\frac{\pi}{2}$, x decreases from 1 to 0 and y increases from 0 to 1; therefore, $\tan\theta = \frac{y}{x}$ increases from 0 to ∞.
2. As θ increases from $\frac{\pi}{2}$ to π, x decreases from 0 to – 1 and y decreases from 1 to 0; therefore, $\tan\theta = \frac{y}{x}$ increases from $-\infty$ to 0.
3. As θ increases from π to $\frac{3\pi}{2}$, x increases from – 1 to 0 and y decreases from 0 to – 1; therefore, $\tan\theta = \frac{y}{x}$ increases from 0 to ∞.
4. As θ increases from $\frac{3\pi}{2}$ to 2π, x increases from 0 to 1 and y increases from – 1 to 0; therefore, $\tan\theta = \frac{y}{x}$ increases from $-\infty$ to 0.

You may want to add some more specific values to this analysis. In any case, we get the following as the graph of the tangent function.



Definition 3.24: (Periodic Function)
A function $y = f(x)$ is called periodic if there exists a number p such that $f(x+p) = f(x)$ for all x in the domain of f. The smallest such number p iscalled the period of the function

A periodic function keeps repeating the same set of $y-values$ over and over again. The graph of a periodic function shows the same basic segment of its graph being repeated. In the case of sine and cosine functions, the period is 2π . The period of the tangent function is π .

Definition 3.25: (Amplitude of a periodic function)
The amplitude of a periodic function $f(x)$ is $$A = \frac{1}{2}[\text{ maximum value of } f(x) - \text{minimum value of } f(x)]$$

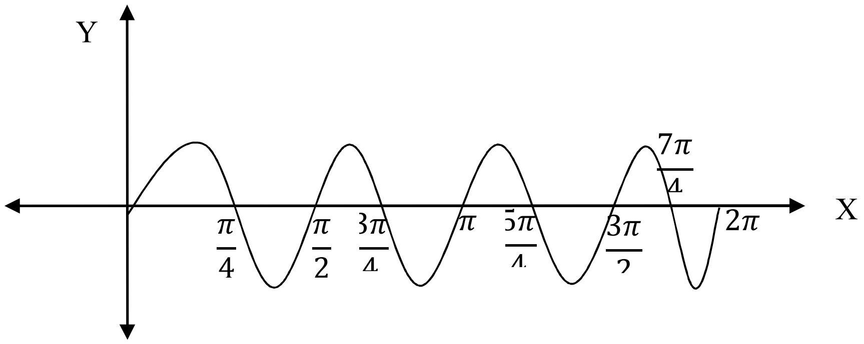
Thus, the amplitude of the basic sine and cosine function is 1.

The portion of the graph of a sine or cosine function over one period is called a complete cycle of the graph. In other words, the minimal portion of a sine or cosine graph that keeps repeating itself is called a complete cycle of the graph.

Definition 3.26: (Frequency of a periodic function)

The number of complete cycles a sine or cosine graph makes on an interval of length equal to 2π is called its frequency.

The frequency of the basic sine curve $y = \sin x$ and the basic cosine curve $y = \cos x$ is 1, because each graph makes 1 complete cycle in the interval $[0, 2\pi]$.
If a sine function has period of $\frac{\pi}{2}$ (see the figure below), then the number of complete cycles its graph will make in an interval of length 2π is $\frac{2\pi}{\pi/2} = 4$.



A sine graph of period $\frac{\pi}{2}$ and frequency 4

Thus if a sine function has a period of $\frac{\pi}{2}$, its frequency is 4 and its graph will make 4 complete cycles in an interval of length 2π.

Example 3.40: Sketch the graph of $y = \sin 2x$ and find its amplitude, period and frequency.

Solution: We can obtain this graph by applying our knowledge of the basic sine graph. For the basic curve, we have

$\sin 0 = 0$ $\quad$ $\sin\frac{\pi}{2} = 1$ $\quad$ $\sin \pi = 0$ $\quad$ $\sin\frac{3\pi}{2} = -1$ $\quad$ $\sin 2\pi = 0$

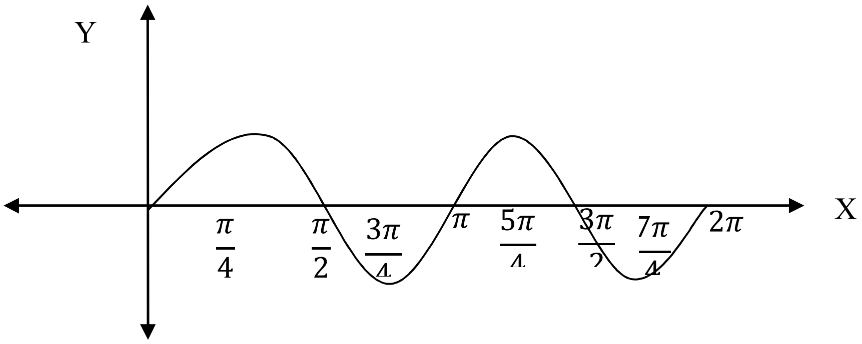
These quadrantal values serve as guide points, which help us draw the graph. To obtain similar guide points for $y = \sin 2x$, we ask for what values of x is

$2x = 0$ $\quad$ $2x = \frac{\pi}{2}$ $\quad$ $2x = \pi$ $\quad$ $2x = \frac{3\pi}{2}$ $\quad$ $2x = 2\pi$

and we get

$x = 0$ $\quad$ $x = \frac{\pi}{4}$ $\quad$ $x = \frac{\pi}{2}$ $\quad$ $x = \frac{3\pi}{2}$ $\quad$ $x = \pi$

Thus, $y = \sin 2x$ will have the values 0, 1, 0, -1, 0 at $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4},$ and π, respectively. The graph of $y = \sin 2x$ will thus complete one cycle in the interval $[0,\pi]$, and will repeat the same values in the interval $[\pi,2\pi]$.



From this graph we see that $y = \sin 2x$ has an amplitude of 1, a period π, and a frequency of 2.

For convenience we summarize our discussion on the domains of the trigonometric functions in the table.

1. $f(x) = \sin x$	Domain = All real numbers
2. $f(x) = \cos x$	Domain = All real numbers
3. $f(x) = \tan x$	Domain = $\{x : x \neq \frac{\pi}{2} + n\pi\}$
4. $f(x) = \csc x$	Domain = $\{ x : x \neq n\pi \}$
5. $f(x) = \sec x$	Domain = $\{x : x \neq \frac{\pi}{2} + n\pi\}$
6. $f(x) = \cot x$	Domain = $\{x : x \neq n\pi\}$
	where n is an integer

In the course of our discussion of the trigonometric functions, we have discussed two types of trigonometric relationships: the reciprocal and quotient relationships. These relationships are examples of trigonometric identities. In the table below we list identities that are satisfied by the trigonometric functions.

The reciprocal Identities	1. $\csc x = \dfrac{1}{\sin x}$
	2. $\sec x = \dfrac{1}{\cos x}$
	3. $\cot x = \dfrac{1}{\tan x}$
The quotient Identities	4. $\tan x = \dfrac{\sin x}{\cos x}$
	5. $\cot x = \dfrac{\cos x}{\sin x}$

The Pythagorean Identities	6. $\sin^2 x + \cos^2 x = 1$ 7. $\tan^2 x + 1 = sce^2 x$ 8. $1 + \cot^2 x = \csc^2 x$
The addition formula	9. (a) $\sin(x+y) = \sin x \cos y + \cos x \sin y$ (b) $\sin(x-y) = \sin x \cos y - \cos x \sin y$ 10. (a) $\cos(x+y) = \cos x \cos y - \sin x \sin y$ (b) $\cos(x-y) = \cos x \cos y + \sin x \sin y$ 11. (a) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ (b) $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$
The double angle formula	12. $\sin 2x = 2 \sin x \cos x$ 13. $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$ 14. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
The half-angle formula	15. $\sin \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$ 16. $\cos \dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$ 17. $\tan \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$

- **Hyperbolic functions and their graphs**

The hyperbolic functions are certain combinations of exponential functions, that occur in various applications, with properties similar to those of the trigonometric functions. Among many other applications they are used to describe the formation of satellite rings around the planets, to describe the shape of a rope hanging from two points, and have application in relativity theory. The two basic hyperbolic functions are the hyperbolic sine and hyperbolic cosine functions. They are defined as follows:

Definition 3.27:	
1. The **hyperbolic sine** function is defined by: $\sinh x = \dfrac{e^x - e^{-x}}{2}$ The domain of $\sinh x$ is $\Re$.	2. The **hyperbolic cosine** function is defined by: $\cosh x = \dfrac{e^x + e^{-x}}{2}$ The domain of $\cosh x$ is also $\Re$.

Remark:

1. $\cosh x$ is pronounced $"kosh"x$ and $\sinh x$ is pronounced as $"cinch"x$.
2. Since $e^{-x} > 0$ for all $x \in \Re$, we see that $\cosh x > \sinh x$ for every $x \in \Re$.
3. If $f(x) = \dfrac{e^x + e^{-x}}{2}$, then $f(-x) = \dfrac{e^{-x} + e^x}{2} = \dfrac{e^x + e^{-x}}{2} = f(x)$. Thus, $\cosh x$ is an even function.
4. *Sinhx* is an odd function.
3. In contrast to sine and cosine, the hyperbolic functions are not periodic.

Example 3.28: Using the above definitions, show that

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\sinh(x+y) = \cosh x \sinh y + \sinh x \cosh y$
3. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Solution:

1. We have

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right) = 1$$

2. $$\sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2} = \frac{e^x e^y - e^{-x}e^{-y}}{2} = \frac{2e^x e^y - 2e^{-x}e^{-y}}{4}$$

$$= \frac{e^x e^y - e^x e^{-y} + e^{-x}e^y - e^{-x}e^{-y}}{4} + \frac{e^x e^y + e^x e^{-y} - e^{-x}e^y - e^{-x-y}}{4}$$

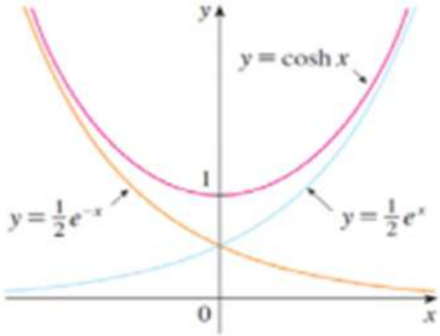
$$= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right)$$

$$= \cosh x \sinh y + \sinh x \cosh y$$

3. Left as an exercise.

- **The graph of** $y = \cosh x$

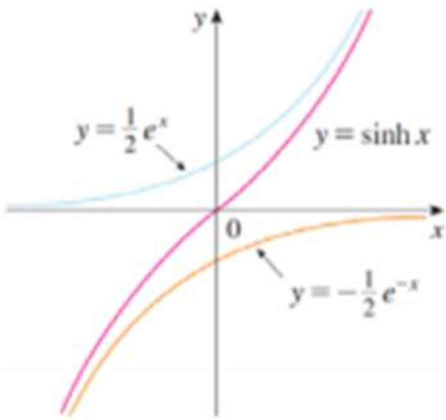
Since $\cosh x$ is an even function, its graph is symmetric about the $y-axis$. Its $y-$ intercept is $(0,1)$, because $\cosh(0) = 1$. As x tends to infinity, $\cosh x = \dfrac{e^x}{2} + \dfrac{e^{-x}}{2}$ tends to infinity because $\dfrac{e^x}{2}$ goes to infinity and $\dfrac{e^{-x}}{2}$ approaches to 0. When x is a large negative number $\cosh x$ acts like $\dfrac{e^{-x}}{2}$, because $\dfrac{e^x}{2}$ gets close to 0. Thus the graph of $y = \cosh x$ looks like:



This graph can also be obtained by geometrically adding the two curves $y = e^x$ and $y = e^{-x}$, and taking half of each resulting $y - value$. Observe that range of $\cosh x$ is $[1, \infty)$.

- **The graph of** $y = \sinh x$

Since $\sinh x$ is an odd function, its graph is symmetric about the origin. The graph passes through the origin because $\sinh(0) = 0$. As x gets large $\sinh x$ acts like $\frac{e^x}{2}$ and when x is a large negative number, $\sinh x$ acts like $-\frac{e^{-x}}{2}$. Thus, the graph of $y = \sinh x$ looks like:



The remaining four hyperbolic functions are defined in terms of $\cosh x$ and $\sinh x$ by analogy with trigonometry.

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	(The domain of $\tanh x$ is $\Re$).
$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	(The domain of $\coth x$ is $\Re \setminus \{0\}$)

$\operatorname{sec}h\,x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$	(The domain of $\operatorname{sec}hx$ is $\Re$)
$\operatorname{csc}h\,x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$	(The domain of $\operatorname{csc}h\,x$ is $\Re \setminus \{0\}$)

You may sketch the graphs of these four hyperbolic functions (see exercise 19).

The trigonometric terminology and notation for the hyperbolic functions stem from the fact that they satisfy a list of identities that much resemble the familiar trigonometric identities, apart from an occasional difference of sign.

$\cosh^2 x - \sinh^2 x = 1$	(1)
$1 - \tanh^2 x = \operatorname{sec}h^2 x$	(2)
$\coth^2 x - 1 = \operatorname{csc}h^2 x$	(3)
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	(4)
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$	(5)

The trigonometric functions are sometimes called circular functions because the point $(\cos\theta, \sin\theta)$ lies on the circle $x^2 + y^2 = 1$ for all θ . Similarly, identity (1) tells us that the point $(\cosh\theta, \sinh\theta)$ lies on the hyperbola $x^2 - y^2 = 1$, and this is the reason for the name hyperbolic functions.

Exercise 3.5

1. Find the domain of the given function.

   a) $f(x) = \dfrac{1}{6^x}$ b) $g(x) = \sqrt{3^x + 1}$ c) $h(x) = \sqrt{2^x - 8}$ d) $f(x) = \dfrac{1}{2^{3x} - 2}$

2. Sketch the graph of the given function. Identify the domain, range, intercepts, and asymptotes.

   a) $y = 5^{-x}$ b) $y = 9 - 3^x$ c) $y = 1 - e^{-x}$ d) $y = e^{x-2}$

3. Solve the given exponential equation.

   a) $2^{x-1} = 8$ b) $3^{2x} = 243$ c) $8^x = \sqrt{2}$ d) $16^{3a-2} = \frac{1}{4}$

4. Let $f(x) = 2^x$. Show that $f(x+3) = 8f(x)$.

5. Let $g(x) = 5^x$. Show that $g(x-2) = \dfrac{1}{25} g(x)$.

6. Let $f(x) = 3^x$. Show that $\dfrac{f(x+2) - f(2)}{2} = 4(3^x)$.

7. Evaluate the given logarithmic expression (where it is defined).

a) $\log_2 32$ c) $\log_3(-9)$ e) $\log_5(\log_3 243)$

b) $\log_{\frac{1}{3}} 9$ d) $\log_6 \frac{1}{\sqrt{6}}$ f) $2^{\log_2 \sqrt{5}}$

8. If $f(x) = \log_2(x^2 - 4)$, find $f(6)$ and the domain of f.

9. If $g(x) = \log_3(x^2 - 4x + 3)$, find $f(4)$ and the domain of g.

10. Show that $\log_{\frac{1}{6}} x = -\log_6 x$

11. Sketch the graph of the given function and identify the domain, range, intercepts and asymptotes.

a) $f(x) = \log_2(x-3)$ b) $f(x) = -3 + \log_2 x$ c) $f(x) = -\log_3(-x)$ d) $f(x) = 3\log_5 x$

12. Find the inverse of $f(x) = e^{(3x-1)}$.

13. Let $f(x) = e^{\sqrt{x}}$. Find a function so that $(f \circ g)(x) = (g \circ f)(x) = x$.

14. Convert the given angle from radians to degrees

a) $\frac{\pi}{3}$ b) $-\frac{5\pi}{2}$ c) $-\frac{4\pi}{3}$

15. Convert the given angle from degrees to radians

a) 315° b) -40° c) 330°

16. Sketch the graph of

a) $f(\theta) = \sec\theta$ c) $f(\theta) = \csc\theta$ e) $f(\theta) = \cot\theta$

b) $f(x) = 1 + \cos x$ d) $f(x) = \sin(x + \frac{\pi}{2})$ f) $f(x) = \tan 2x$

17. Verify the following identities:

a) $(\sin x - \cos x)(\csc x + \sec x) = \tan x - \cot x$

b) $\sec^2 x - \csc^2 x = \tan^2 x - \cot^2 x$

18. Given $\tan\theta = \frac{1}{2}$ and $\sin\theta < 0$, find $\cos\theta$.

19. Sketch the graphs of

a) $f(x) = \tanh x$ c) $f(x) = \csc hx$

b) $f(x) = \sec hx$ d) $f(x) = \coth x$

20. Prove the identities (2) and (3).

21. Find the exact numerical value of

a) $\sinh(\ln 2)$ b) $\cosh(-\ln 3)$ c) $\tanh(2\ln 3)$

22. Prove the following identities:

a) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

b) $\cosh y(x-y) = \cosh x \cosh y - \sinh x \sinh y$

Chapter 4: Analytic Geometry

The main topics of study in analytic geometry are straight lines and conic sections. Accordingly, by the end of this chapter you must

- be able to derive basic equations that are representing straight lines, circles, parabolas, ellipses, and hyperbola.
- know the main (important) properties of each of these five geometric objects.
- be able to identify equations of straight lines, circles, parabolas, ellipses, hyperbolas and sketch their graphs.

More specific objectives are given in each section.

The major part of this chapter is **conic sections**. Conic sections are **circles**, **parabolas, ellipse**s and **hyperbolas**. They are called conic sections because they are generated when a plane cuts a right circular double cone. Depending on how the plane cuts the cone the intersection forms a curve called a circle, an ellipse, a parabola or a hyperbola (See, Figure 4.1).

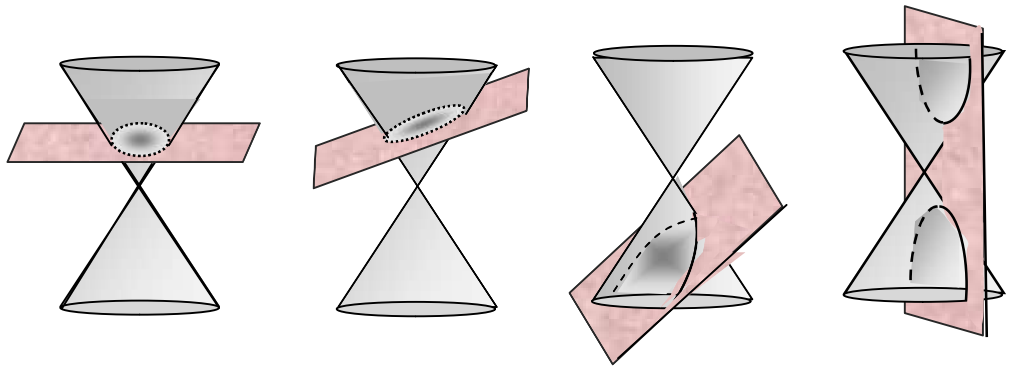


Figure 4.1: (a) circle (b) ellipse (c) parabola (d) hyperbola

We will see that a conic section is described by a second degree equation in x and y of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

when A, C, D, E and F are constant real numbers. In the analysis of such equations we will frequently need the method of completing the square. Recall that completing the square is the method of converting an equation of the form

$x^2 + ax = b$ to $(x+h)^2 = c$ (*Can you establish the relationships between a,b and h,c ?*)

To do this :- *Add* $\left(\frac{a}{2}\right)^2$ *to both sides of the former equation.*

- *Then complete the square of the resulting expression to get the later form.*

Here recall that: $x^2 + 2ax + a^2 = (x+a)^2$ and $x^2 - 2ax + a^2 = (x-a)^2$.

4.1 Distance Formula and Equation of Lines

By the end of this section, you should

- be able to find the distance between two points in the coordinate plane.
- be able to find the coordinates of a point that divides a line segment in a given ratio.
- know different forms of basic equations of a line
- be able to find equation of a line and draw the line.
- know when two lines are parallel.
- know when two lines are perpendicular.
- be able to find the distance between a point and a line in the coordinate plane.

4.1.1 Distance between two points and division of segments

If P and Q are two points on the coordinate plane, then PQ represents the line segment joining P and Q and $d(P,Q)$ or $|PQ|$ represents the distance between P and Q.

Recall that the distance between points a and b on a number line is $|a - b| = |b - a|$. Thus, the distance between two points $P(x_1, y_1)$ and $R(x_2, y_1)$ on a horizontal line must be $|x_2 - x_1|$ and the distance between $Q(x_2, y_2)$ and $R(x_2, y_1)$ on a vertical line must be $|y_2 - y_1|$. (See Figure 4.2).

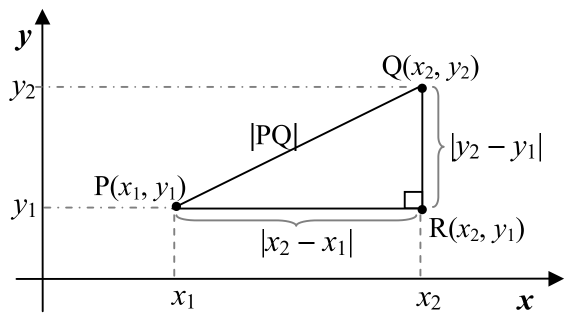


Figure 4.2

To find distance $|PQ|$ between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, we note that triangle PRQ in Figure 4.2 is a right triangle, and so by Pythagorean Theorem we get:

$$|PQ|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 \Leftrightarrow |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore, we have the following:

Distance Formula: The distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note that, from the distance formula, the distance between the origin O(0,0) and a point P(x, y) is

$$|OP| = \sqrt{x^2 + y^2}$$

Example 4.1: (i) The distance between O(0,0) and P(3,4) is

$$|OP| = \sqrt{3^2 + 4^2} = 5.$$

(ii) The distance between P(1,2) and Q(3,6) is

$$|PQ| = \sqrt{(3-1)^2 + (6-2)^2} = \sqrt{20}.$$

(iii) The distance between P(−1,2) and Q(5,−6) is

$$|PQ| = \sqrt{(5+1)^2 + (-6-2)^2} = 10.$$

Division point of a line segment: Given two distinct points P(x_1, y_1) and Q(x_2, y_2) in the coordinate plane, we want to find the coordinates (x_0, y_0) of the point R that lies on the segment PQ and divides the segment in the ratio r_1 to r_2; that is

$$\frac{|PR|}{|RQ|} = \frac{r_1}{r_2},$$

where r_1 and r_2 are given positive numbers. (See Figure 4.3).

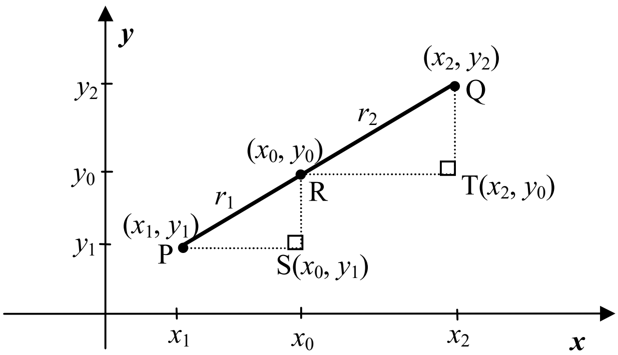


Figure 4.3

To determine (x_0, y_0), we construct two right triangles ΔPSR and ΔRTQ as in Figure 4.3. We then have |PS| = $x_0 - x_1$, |SR| = $y_0 - y_1$, |RT| = $x_2 - x_0$, and |TQ| = $y_2 - y_0$. Now since ΔPSR is similar to ΔRTQ, we have that

$$\frac{x_0 - x_1}{x_2 - x_0} = \frac{r_1}{r_2} \quad \text{and} \quad \frac{y_0 - y_1}{y_2 - y_0} = \frac{r_1}{r_2}$$

or $r_2(x_0 - x_1) = r_1(x_2 - x_0)$ and $r_2(y_0 - y_1) = r_1(y_2 - y_0)$.

Solving for x_0 and y_0, we obtain $x_0 = \dfrac{x_1 r_2 + x_2 r_1}{r_1 + r_2}$ and $y_0 = \dfrac{y_1 r_2 + y_2 r_1}{r_1 + r_2}$

Therefore, we have shown the following.

Theorem 4.1: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be distinct points in the coordinate plane. If $R(x_0, y_0)$ is a point on the line segment PQ that divides the segment in the ratio $|PR| : |RQ| = r_1 : r_2$, then the coordinates of R is given by

$$(x_0, y_0) = \left(\frac{x_1 r_2 + x_2 r_1}{r_1 + r_2}, \frac{y_1 r_2 + y_2 r_1}{r_1 + r_2}\right)$$

In particular, the midpoint of PQ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Example 4.2: Given P(–3, 3) and Q(7,8),

(i) find the coordinates of the point R on the line segment PQ such that $|PR| : |RQ| = 2 : 3$.

(ii) find the coordinates of the midpoint of PQ.

Solution: (i) Obviously $R(x_0, y_0)$ is given by

$$(x_0, y_0) = \left(\frac{-3 \times 3 + 7 \times 2}{2 + 3}, \frac{3 \times 3 + 8 \times 2}{2 + 3}\right) = (1, 5)$$

(ii) The coordinates of the midpoint is $\left(\frac{-3 + 7}{2}, \frac{3 + 8}{2}\right) = (2,\ 1\,1/2\ )$.

Exercise 4.1.1

1. Find the distance between the following pair of points.
   (a) (–1, 0) and (3, 0).
   (b) (1, –2) and (1, 4).
   (c) (–2, 3) and (2, 0)
   (d) The origin and $(-\sqrt{3}, \sqrt{6})$.
   (e) (a, a) and $(-a, -a)$
   (f) (a, b) and $(-a, -b)$
2. If the vertices of ΔABC are A(1,1), B(4,5) and C(7, 1), find the perimeter of the triangle.
3. Let P = (–3,0) and Q be a point on the positive y-axis. Find the coordinates of Q if |PQ| =5.
4. Suppose the endpoints of a line segment AB are A(–1,1) and B(5, 10). Find the coordinates of point P and Q if
   (a) P is the midpoint of AB.
   (b) P divides AB in the ratio 2:3 (That is, |AP|: |PB| = 2:3).
   (c) Q divides AB in the ratio 3:2.
   (d) P and Q trisect AB (i.e., divide it into three equal parts).
5. Let M(–1,3) be the midpoint of a line segment PQ. If the coordinates of P is (–5, –7), then what is the coordinates of Q?
6. Let A(a, 0), B(0,b) and O(0,0) be the vertices of a right triangle. Show that the midpoint of AB is equidistant from the vertices of the triangle

4.1.2 Equations of lines

An equation of a line ***l*** is an equation which must be satisfied by the coordinates (x, y) of every point on the line. A line can be vertical, horizontal or oblique. The equation of a vertical line that intersects the x-axis at $(a, 0)$ is $x=a$ because the x-coordinate of every point on the line is a. Similarly, the equation of a horizontal line that intersects the y-axis at $(0, b)$ is $y=b$ because the y-coordinate of every point on the line is b.

An oblique line is a straight line which is neither vertical nor horizontal. To find equation of an oblique line we use its slope which is the measure of the steepness of the line. In particular, the slope of a line is defined as follows.

Definition 4.1. The **slope** of a non-vertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined. Note that the slope of horizontal line is 0.

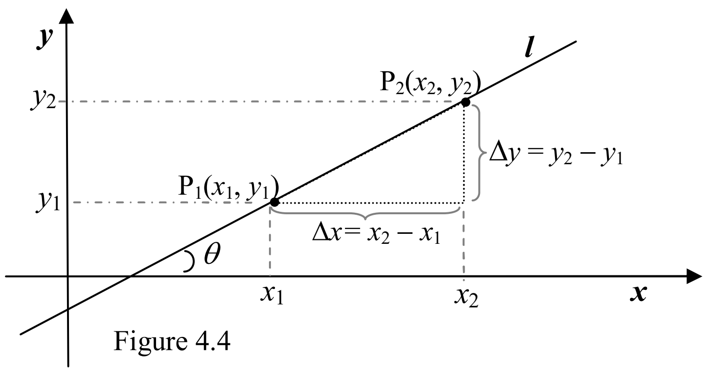


Figure 4.4

Thus the slope of a line ***l*** is the ratio of the change in y, Δy, to the change in x, Δx (see Figure 4.4). Hence, slope is the rate of change of y with respect x. The slope depends also on the angle of inclination of the line. Note that the angle of inclination θ is the angle between x-axis and the line (measured counterclockwise from the direction of positive x-axis to the line). Observe that

$$\tan\theta = \frac{\Delta y}{\Delta x}$$

Therefore, if θ is the angle of inclination of a line, then its slope is $m = \tan\theta$.

Now let us find an equation of the line that passes through a point $P_1(x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line though P_1 and P is m; that is

$$\frac{y - y_1}{x - x_1} = m$$

This leads to the following equation of the line:

$$y - y_1 = m(x - x_1)$$ (called point-slope form of equation of a line).

In general, depending on the given information, you can show that the equations of oblique lines can be obtained using the following formulas.

Given Information	Formula for Equation of the Line
Slope m and its y-intercept $(0,b)$	Slope-Intercept-Form: $y = mx + b$
Slope m and a point (x_1, y_1) on $\boldsymbol{l}$	Point-Slope-Form: $y - y_1 = m(x - x_1)$ Or $y = m(x - x_1) + y_1$
Two points (x_1, y_1) and (x_2, y_2) on $\boldsymbol{l}$	Two-Point Form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
x-intercept $(a,0)$ and y-intercept $(0,b)$	Intercept Form: $\frac{x}{a} + \frac{y}{b} = 1$

Example 4.3: Find an equation of the line $\boldsymbol{l}$ if

(i) the line passes through $(3, -2)$ and its angle of inclination is 135°.

(ii) the line passes through the points $(1, 2)$ and $(4, -2)$

Solution: (i) The slope of $\boldsymbol{l}$ is $m = \tan(135°) = -1$; and it passes through point $(3, -2)$. Thus, using the point-slope form with $x_1 = 3$ and $y_1 = -2$, we obtain the equation of the line as $y - (-2) = -1(x - 3)$ which simplifies to $y = -x + 1$.

(ii) Given the line passes through $(1, 2)$ and $(4, -2)$, the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 2}{4 - 1} = \frac{-4}{3}.$$

So, using the point-slope form with $x_1 = 1$ and $y_1 = 2$, we obtain the equation of the line as $y - 2 = \frac{-4}{3}(x - 1)$ which simplifies to $4x + 3y = 10$.

(Note that it is possible to use the two-point form to find the equation of this line)

General Form: In general, the equation of a straight line can be written as

$$ax + by + c = 0,$$

for constants a, b, c with a and b not both zero. Indeed, if $a=0$ the line is a horizontal line given by $y = -c/b$, if $b=0$ the line is a vertical line given by $x = -c/a$, and if both $a, b \neq 0$ it is the oblique line given by $y = -(a/b)x - c/b$ with slope $m = -a/b$ and y-intercept $-c/b$.

Parallel and Perpendicular lines: slopes can be used to check whether lines are parallel, perpendicular or not. In particular, let $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ be non-vertical lines with slope m_1 and m_2, respectively. Then,

. (i) $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are parallel, denoted by $\boldsymbol{l}_1 \parallel \boldsymbol{l}_2$, iff $m_1 = m_2$.

(ii) $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are perpendicular, denoted by $\boldsymbol{l}_1 \perp \boldsymbol{l}_2$, iff $m_1 m_2 = -1$ (or $m_2 = -\dfrac{1}{m_1}$)

Moreover, if $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are both vertical lines then they are parallel. However, if one of them is horizontal and the other is vertical, then they are perpendicular.

Example 4.4: Find an equation of the line through the point (3,2) that is parallel to the line $2x + 3y + 5 = 0$.

Solution: The given line can be written in the form $y = \frac{-2}{3}x - \frac{5}{3}$ which is the slope-intercept form; that is, it has slope $m = -2/3$. So, as parallel lines have the same slope, the required line has slope $-2/3$. Therefore, its equation in point-slope form is $y - 2 = \frac{-2}{3}(x - 3)$ which can be simplified to $2x + 3y = 12$.

Example 4.5: Show that the lines $2x + 3y + 5 = 0$ and $3x - 2y - 4 = 0$ are perpendicular.

Solution: The equations can be written as $y = \frac{-2}{3}x - \frac{5}{3}$ and $y = \frac{3}{2}x - 2$ from which we can see that $m_1 = -2/3$ and $m_2 = 3/2$. Since $m_1 m_2 = -1$, the lines are perpendicular.

Exercise 4.1.2

1. Find the slope and equation of the line determined by the following pair of points. Also find the y- and x- intercepts, if any, and draw each line.

(a) (0, 2) and (3, 2) (e) The origin and (1,2) (i) (−1, 3) and (1, 6)

(b) (2, 0) and (2, 3) (f) The origin and (1,−3) (j) (−3, −2) and (2, −2)

(c) The origin and (1,0) (g) (1,2) and (3, 4) (k) (0, 3) and (3, 0)

(d) The origin and (−1, 0) (h) (−2, −3), (2, 5) (l) (−1, 0) and (0, 2)

2. Find the slope and equation of the line whose angle of inclination is θ and passes through the point P, if

(a) $\theta = \frac{1}{4}\pi$, P = (1,1). (d) $\theta = 0$, P = (0, 1).

(b) $\theta = \frac{1}{4}\pi$, P = (0,1). (e) $\theta = \frac{1}{3}\pi$, P = (1, 3).

(c) $\theta = \frac{3}{4}\pi$, P = (0,1). (f) $\theta = \frac{1}{3}\pi$, P = (1,–3).

3. Find the $\boldsymbol{x}$-and $\boldsymbol{y}$-intercepts and slope of the line given by $\frac{x}{2} - \frac{y}{3} = 1$, and draw the line.

4. Draw the triangle with vertices A(−2,4), B(1,−1) and C(6,2) and find the following.
   (a) Equations of the sides.
   (b) Equations of the medians.
   (c) Equations of the perpendicular bisectors of the sides.
   (d) Equation of the lines through the vertices parallel to the opposite sides.
5. Find the equation of the line that passes through (2, −1) and perpendicular to $3x + 4y = 6$.
6. Suppose ℓ_1 and ℓ_2 are perpendicular lines intersecting at (−1, 2). If the angle of inclination of ℓ_1 is 45°, then find an equation of ℓ_2.
7. Determine which of the following pair of lines are parallel, perpendicular or neither.
   (a) $2x - y + 1 = 0$ and $2x + 4y = 3$
   (b) $3x - 6y + 1 = 0$ and $x - 2y = 3$
   (c) $2x + 5y + 3 = 0$ and $5x + 3y + 2 = 0$
   (d) $y = 3x + 2$ and $3x + y = 2$
   (e) $2x - 3y = 5$ and $3x + 2y - 3 = 0$
   (f) $\frac{x}{3} + \frac{y}{2} = 1$ and $2x + 3y - 6 = 0$
8. Let L_1 be the line passing through P(a, b) and Q(b, a) such that $a \neq b$. Find an equation of the line L_2 in terms of a and b if
   (a) L_2 passes through P and perpendicular to L_1.
   (b) L_2 passes through (a, a) and parallel to L_1.
9. Let L_1 and L_2 be given by $2x + 3y - 4 = 0$ and $x + 3y - 5 = 0$, respectively. A third line L_3 is perpendicular to L_1. Find an equation of L_3 if the three lines intersect at the same point.
10. Determine the value(s) of k for which the line

$$(k+2)x + (k^2 - 9)y + 3k^2 - 8k + 5 = 0$$

   (a) is parallel to the x-axis.
   (b) is parallel to the y-axis.
   (c) passes through the origin
   (d) passes through the point (1,1).

   In each case write the equation of the line.
11. Determine the values of a and b for which the two lines $ax - 2y = 1$ and $6x - 4y = b$
   (a) have exactly one intersection point.
   (b) are distinct parallel lines.
   (c) coincide.
   (d) are perpendicular.

4.1.3 Distance between a point and a line

Suppose a line $\boldsymbol{l}$ and a point P(x,y) not on the line are given. The distance from P to $\boldsymbol{l}$, d(P, $\boldsymbol{l}$), is defined as the perpendicular distance between P and $\boldsymbol{l}$. That is,

d(P, $\boldsymbol{l}$) = |PQ|, where Q is the point on $\boldsymbol{l}$ such that PQ$\perp$ $\boldsymbol{l}$. (See Figure 4.5)

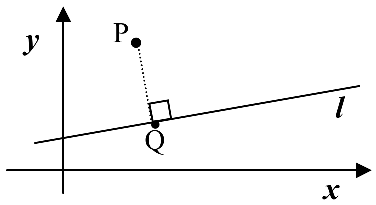


Figure 4.5: |PQ| = d(P, $\boldsymbol{l}$)

If P is on $\boldsymbol{l}$, then d(P, $\boldsymbol{l}$) = 0. Moreover, given a point P(h,k) observe that

(i) if the line $\boldsymbol{l}$ is a horizontal line $y=b$, then d(P, $\boldsymbol{l}$) = $|k - b|$.

(ii) if the line $\boldsymbol{l}$ is a vertical line $x=a$, then d(P, $\boldsymbol{l}$) = $|h - a|$

In general, however, to find the distance between a point P(x_0, y_0) and an arbitrary line $\boldsymbol{l}$ given by $ax + by + c = 0$, we have to first get a point Q on $\boldsymbol{l}$ such that PQ$\perp$ $\boldsymbol{l}$ and then compute |PQ|. This yields the formula given in the following Theorem.

Theorem 4.2: The distance between a point P(x_0, y_0) and a line L: $ax + by + c = 0$ is given by

$$d(\mathrm{P}, L) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

In particular, if we take $(x_0,y_0)=(0,0)$ in this formula, we obtain the distance between the origin O(0,0) and a line L : $ax + by + c = 0$ which is given by

$$d(\mathrm{O}, L) = \frac{|c|}{\sqrt{a^2 + b^2}}$$

Example 4.6: Show that the origin and P(6,4) are equidistant from the line L: $y = -(3/2)x + 13/2$.

Solution: By equidistant we mean equal distance. So, we need to show d(O, L) = d(P, L).

To use the above formula, we first write the equation of the line L in the general form which is $3x + 2y - 13 = 0$. Thus, $a = 3$, $b = 2$ and $c = -13$.

$$\Rightarrow \quad d(\mathrm{O}, L) = \frac{|c|}{\sqrt{a^2 + b^2}} = \frac{|-13|}{\sqrt{9+4}} = \frac{13}{\sqrt{13}}$$

and $$d(\mathrm{P}, L) = \frac{|3 \times 6 + 2 \times 4 - 13|}{\sqrt{3^2 + 2^2}} = \frac{13}{\sqrt{13}}$$

Therefore, d(O, L) = d(P, L) = $13/\sqrt{13}$.

Thus, O(0,0) and P(6,4) are equidistant from the given line L.

Exercise 4.1.3

1. Find the distance between the line L given by $y = 2x + 3$ and each of the following points.
   (a) The origin (b) (2, 3) (c) (1, 5) (d) (−1, −1)
2. Suppose L is the line through (1, 2) and (3, 2). What is the distance between L and
   (a) The origin (b) (2, −3) (c) (a, 0) (d) (a, b) (e) (a, 2)
3. Suppose L is the vertical line that crosses the $\boldsymbol{x}$-axis at (5, 0). Find $d(P, L)$, when P is
   (a) The origin (b) (2, −4) (c) (0, b) (d) (5, b) (e) (a, b)
4. Suppose L is the line that passes through (0, −3) and (4, 0). Find the distance between L and each of the following points.
   (a) The origin (b) (1, 4) (c) (−1, 0) (d) (8, 3)
   (e) (0, 1) (f) (4, −2) (g) (1, −9/4) (h) (7, −4)
5. The vertices of ΔABC are given below. Find the length of the side BC, the height of the altitude from vertex A to BC, and the area of the triangle when its vertices are
   (a) A(3, 4), B(2, 1), and C(6, 1).
   (b) A(3, 4), B(1, 1), and C(5, 2).
6. Consider the quadrilateral whose vertices are A(1,2), B(2,6), C(6,8) and D(5,4). Then,
   (a) Show that the quadrilateral is a parallelogram.
   (b) How long is the side AD?
   (c) What is the height of the altitude of the quadrilateral from vertex A to the side AD.
   (d) Determine the area of the quadrilateral.

4.2 Circles

By the end of this section, you should

- know the geometric definition of a circle.
- be able to identify whether a given point is on, inside or outside a circle.
- be able to construct equation of a circle.
- be able to identify equations that represent circles
- be able to find the center and radius of a circle and sketch its graph if its equation is given.
- be able to identify whether a given circle and a line intersect at two points, one points or never intersect at all.
- know the properties of a tangent line to a circle.
- be able to find equation of a tangent line to a circle.

4.2.1 Definition of a Circle

Definition 4.2. A circle is the locus of points (set of points) in a plane each of which is equidistant from a fixed point in the plane. The fixed point is called the **center** of the circle and the constant distance is called its **radius**.

Definition 4.2 is illustrated by Figure 4.6 in which the center of the circle is denoted by "C" and its radius is denoted by *r*.

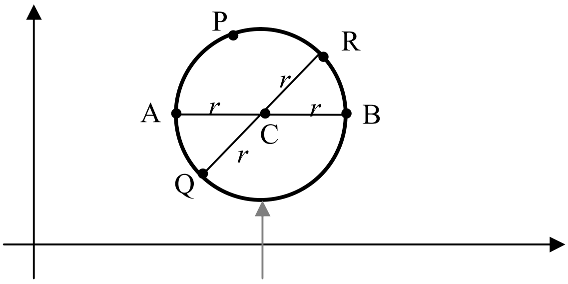


Figure 4.6. Circle with center C, radius *r*

Observe that a circle is symmetric with respect its center. Based on the definition, a point P is on the circle if and only if its distance to C is *r*, that is $|CP| = r$. A point in the plane is said to be inside the circle if its distance to the center C is less than *r*. Similarly, a point in the plane is said to be outside the circle if its distance to C is greater than *r*. Moreover, a chord of the circle is a line segment whose endpoints are on the circle. A diameter is a chord of the circle through the center C. Consequently, C is the midpoint of a diameter and the length of a diameter is $2r$. For example, AB and QR are diameters of the circle in Figure 4.6.

Example 4.7: Consider a circle of radius 5 whose center is at C(2,1). Determine whether each of the following points is on the circle, inside the circle or outside the circle:
$P_1(5, 5)$, $P_2(4, 5)$, $P_3(-2, 5)$, $P_4(-1, -2)$, $P_5(2,-4)$, $P_6(7, 0)$.

Solution: The distance between a given point $P(x,y)$ and the center C(2,1) is given by
$|PC| = \sqrt{(x-2)^2 + (y-1)^2}$ or $|PC|^2 = (x-2)^2 + (y-1)^2$. We need to compare $|PC|$ with the radius 5. Note that $|PC| = 5 \Leftrightarrow |PC|^2 = 25$, $|PC| < 5 \Leftrightarrow |PC|^2 < 25$,
and $|PC| > 5 \Leftrightarrow |PC|^2 > 25$.

Thus, P is on the circle if $|PC|^2 = 25$, inside the circle if $|PC|^2 < 25$ and outside the circle if $|PC|^2 > 25$. So, we can use the square distance to answer the question. Thus, as
$|P_1C|^2 = (5-2)^2+(5-1)^2 = 25$, $|P_2C|^2 = (4-2)^2+(5-1)^2 = 20$ and $|P_3C|^2 = (-2-2)^2+(5-1)^2 = 32$,
P_1 is on the circle, P_2 is inside the circle, and P_3 is outside the circle. Similarly, you can show that P_4 is inside the circle, P_5 is on the circle, and P_6 is outside the circle.

Exercise 4.2.1

1. Suppose the center of a circle is C(1,–2) and P(7, 6) is a point on the circle. What is the radius of the circle?
2. Let A(1, 2) and B(5, −2) are endpoints of a diameter of a circle. Find the center and radius of the circle.
3. Consider a circle whose center is the origin and radius is $\sqrt{5}$. Determine whether or not the circle contains the following point.

   (a) (1, 2) (b) (0,0) (c) $(0,-\sqrt{5})$ (d) (3/2, 3/2)

   (e) (5, 0) (f) (−1, −2), (g) $(\sqrt{3},\sqrt{2})$ (h) (5/2, 5/2)
4. Consider a circle of radius 5 whose center is at C(−3,4). Determine whether each of the following points is on the circle, inside the circle or outside the circle:

   (a) (0, 9) (b) (0,0) (c) (1,6) (d) (1, 0)

   (e) (−7, 1) (f) (−1, −1), (g) (2,4) (h) (5/2, 5/2)

4.2.2 Equation of a Circle

We now construct an equation that the coordinates (x,y) of the points on the circle should satisfy. So, let P(x,y) be any point on a circle of radius r and center C(h,k) (see, Figure 4.7). Then, the definition of a circle requires that

$$|CP| = r$$

$$\Rightarrow \quad \sqrt{(x-h)^2+(y-k)^2} = r$$

or

$$(x-h)^2+(y-k)^2 = r^2$$

(Standard equation of a circle with center (h,k) and radius r.

In particular, if the center is at origin, i.e., $(h,k) = (0,0)$, the equation is

$$x^2+y^2 = r^2$$

(Standard Equation of a circle of radius r centered at origin)

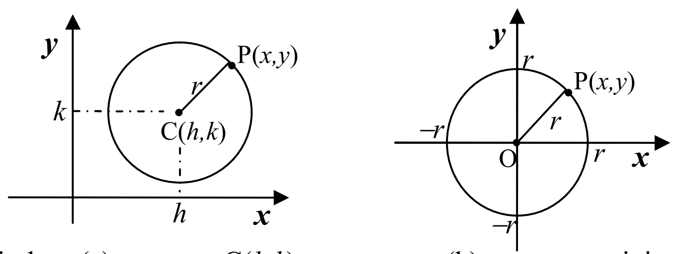


Figure 4.7 circles (a) center at C(h,k) (b) center at origin

Example 4.8: Find an equation of the circle with radius 4 and center (–2, 1).

Solution: Using the standard equation of a circle in which the center $(h, k) = (-2, 1)$ and radius $r = 4$ we obtain the equation

$$(x+2)^2 + (y-1)^2 = 16.$$

Example 4.9: Find the equation of a circle with endpoints of a diameter at P(–2, 0) and Q(4, 2).

Solution: The center of the circle C(*h,k*) is the mid-point of the diameter. Hence, $(h,k) = \left(\frac{-2+4}{2}, \frac{0+2}{2}\right) = (1,1)$. Also, for its radius *r*, $r^2 = |CP|^2 = (1+2)^2 + (1-0)^2 = 10$.

Thus, the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$. That is,

$$(x-1)^2 + (y-1)^2 = 10.$$

Example 4.10: Suppose P(–2,4) and Q(5,3) are points on a circle whose center is on *x*-axis. Find the equation of the circle.

Solution: We need to obtain the center C and radius *r* of the circle to construct its equation. As the center is on *x*-axis, its second coordinate is 0. Therefore, let C(*h*,0) be the center of the circle. Note that $|PC|^2 = |QC|^2 = r^2$ as both P and Q are on the circle. So, from the first equality we get $(-2-h)^2 + 4^2 = (5-h)^2 + 3^2$. Solving this for *h* we get *h*=1. Hence, the center is at C(1, 0) and $r^2 = |QC|^2 = (5-1)^2 + 3^2 = 25$. Therefore, the equation of the circle is $(x-1)^2 + y^2 = 25$.

Example 4.11: Determine whether the given equation represents a circle. If it does, identify its center and radius and sketch its graph.

(a) $x^2 + y^2 + 2x - 6y + 7 = 0$

(b) $x^2 + y^2 + 2x - 6y + 10 = 0$

(c) $x^2 + y^2 + 2x - 6y + 11 = 0$

Solution: We need to rewrite each equation in standard form to identify its center and radius. We do this by completing the square on the x-terms and y-terms of the equation as follows:

(a) $(x^2 + 2x) + (y^2 - 6y) = -7.$ (Grouping x-terms and y-terms)

$\Leftrightarrow (x^2 + 2x + 1^2) + (y^2 - 6y + 3^2) = -7 + 1 + 9.$ (Adding 1^2 and 3^2 to both sides)

$\Leftrightarrow (x+1)^2 + (y-3)^2 = 3.$

Comparing this with the standard equation of circle this is equation of a circle with center $(h, k) = (-1, 3)$ and radius $r = \sqrt{3}$. The graph of the circle is sketched in Figure 4.8

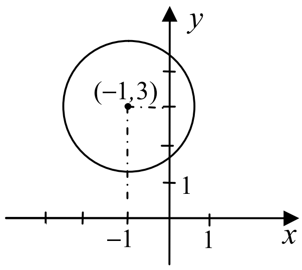


Figure 4. 8

(b) Following the same steps as in (a), you can see that $x^2 + y^2 + 2x - 6y + 10 = 0$ is equivalent to $(x + 1)^2 + (y - 3)^2 = 0$.
This is satisfied by the point (–1, 3) only. The locus of this equation is considered as a point-circle, circle of zero radius (sometimes called degenerated circle).

(c) Again following the same steps as in (a), you can see that $x^2 + y^2 + 2x - 6y + 11 = 0$ is equivalent to $(x + 1)^2 + (y - 3)^2 = -1$.
Note that this does not represent a circle; in fact it has no locus at all (Why?).

Remark: Consider an equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0.$$

By completing the square you can show the following:

- If $D^2 + E^2 - 4F > 0$, then the equation represents a circle with center $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ and radius $r = \frac{1}{2}\sqrt{D^2+E^2-4F}$.
- If $D^2 + E^2 - 4F = 0$, then the equation is satisfied by the point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$ only. In this case the locus of the equation is called **point-circle** (circle of zero radius).
- If $D^2 + E^2 - 4F < 0$, then the equation has no locus.

Exercise 4.2.2

1. Determine whether each of the following points is inside, outside or on the circle with equation $x^2 + y^2 = 5$.
   (a) (–1, 2), (b) (3/2, 2) (c) $(0, -\sqrt{5})$ (d) (–1, 3/2)
2. Find an equation of the circle whose endpoints of a diameter are (0, –3) and (3, 3).
3. Determine an equation of a circle whose center is on *y*-axis and radius is 2.
4. Find an equation of the circle passing through (1, 0) and (0, 1) which has its center on the line $2x + 2y = 5$.
5. Find the value(s) of *k* for which the equation $2x^2 + 2y^2 + 6x - 4y + k = 0$ represent a circle.

6. An equation of a circle is $x^2 + y^2 - 6y + k = 0$. If the radius of the circle is 2, then what is the coordinates of its center?
7. Find equation of the circle passing through (0,0), (4, 0) and (2, 2).
8. Find equation of the circle inscribed in the triangle with vertices (–7, –10), (–7, 15), and (5,–1).
9. In each of the following, check whether or not the given equation represents a circle. If the equation represents a circle, then identify its center and the length of its diameter.

   (a) $x^2 + y^2 - 18x + 24y = 0$ (d) $5x^2 + 5y^2 + 125x + 60y - 100 = 0$

   (b) $x^2 + y^2 - 2x + 4y + 5 = 0$ (e) $36x^2 + 36y^2 + 12x + 24y - 139 = 0$

   (c) $x^2 + y^2 - 4x - 2y + 11 = 0$ (f) $3x^2 + 3y^2 + 2x + 4y + 6 = 0$
10. Show that $x^2 + y^2 + Dx + Ey + F = 0$ represents a circle of positive radius iff $D^2 + E^2 - 4F > 0$.

4.2.3 Intersection of a circle with a line and tangent line to a circle

The number of intersection points of a given line and a circle is at most two; that is, either no intersection point, or only one intersection point, or two intersection points. For instance, in Figure 4.9, the line l_1 has no intersection with the circle, l_2 has two intersection points with the circle, namely, Q_1 and Q_2, and l_3 has only one intersection point with the circle, namely, P.

A line which intersects a circle at one and only one point is called a **tangent line** to the circle. In this case, the intersection point is called the **point of tangency**. Thus, l_3 a tangent line to the circle in Figure 4.9 and P is the point of tangency.

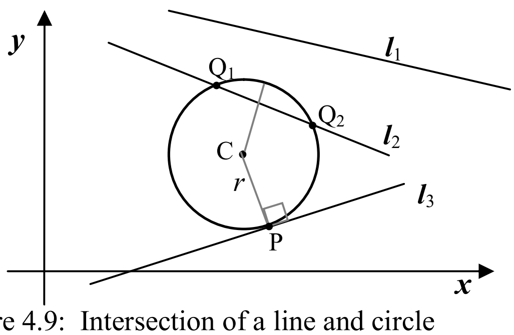


Figure 4.9: Intersection of a line and circle

In Figure 4.9, observe that every point on l_1 are outside of the circle. Hence, $d(C,Q) > r$ for every point Q on l_1. Consequently, $d(C, l_1) > r$. On the other hand, there is a point on l_2 which is inside the circle. Hence, $d(C, l_2) < r$.

For the tangent line l_3, the point of tangency P is on the circle implies that |CP| = r and P is the point on l_3 closest to C. Therefore, $d(C, l_3) = |CP| = r$. This shows also that CP⊥ l_3.

In general, given a circle of radius r with center C(h,k) and a line l, by computing the distance $d(C, l)$ between C and l we can conclude the following.

(i). If $d(C, l) > r$, then the line does not intersect with the circle.

(ii) If $d(C, l) < r$, then the line is a secant of the circle; that is, they have two intersection points.

(iii) If $d(C, l) = r$, then l is a tangent line to the circle. The point of tangency is the point P on the line (and on the circle) such that CP ⊥ l. This means the product of the slopes of l and CP must be −1.

Example 4.12 Write the equation of the circle tangent to the x-axis at (6,0) whose center is on the line $x - 2y = 0$.

Solution: The circle in the question is as in Figure 4.10.

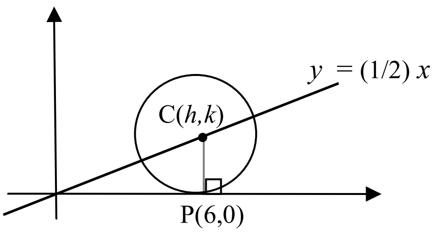


Figure 4.10

Let C(h, k) be the center of the circle. (h, k) is on the line $y = (1/2)x \Rightarrow k = (1/2)h$; and the circle is tangent to x-axis at P(6,0) ⇒ CP should be perpendicular to the x-axis.

$\Rightarrow h = 6 \Rightarrow k = 3$ and the radius is $r = |CP| = k - 0 = 3$.

Hence, the circle is centered at (6, 3) with radius $r = 3$. Therefore, the equation of the circle is $(x-6)^2 + (y-3)^2 = 9$.

Example 4.13 Suppose the line $y=x$ is tangent to a circle at point P(2,2). If the center of the circle is on the x-axis, then what is the equation of the circle?

Solution: The circle in the question is as in Figure 4.11.

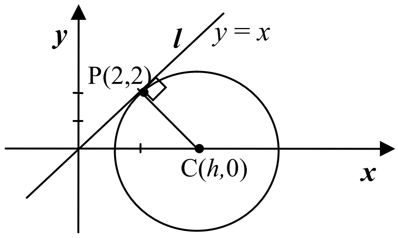


Figure 4.11

Let the center of the circle be C(h,0). We need to find h. The slope of the line $\boldsymbol{l}$: $y=x$ is 1 and $\boldsymbol{l}$ is perpendicular to CP. Hence the slope of CP is –1.

So, the slope of CP = $\frac{2-0}{2-h}=-1 \Rightarrow h-2=2$ or $h=4$.

$\Rightarrow$ The center of the circle is C(4,0); and $r^2 = |CP|^2 = (2-0)^2 + (2-4)^2 = 4+4 = 8$.

Therefore, the equation of the circle is $(x-4)^2 + y^2 = r^2 = 8$.

Exercise 4.2.3

1. Find the equation of the line tangent to the circle with the center at (–1, 1) and point of tangency at (–1, 3).
2. The center of a circle is on the line $y = 2x$ and the line $x=1$ is tangent to the circle at (1, 6). Find the center and radius of the circle.
3. Suppose two lines $y = x$ and $y = x - 4$ are tangent to a circle at (2, 2) and (4, 0), respectively. Find equation of the circle.
4. Find an equation of the line tangent to the circle $x^2 + y^2 - 2x + 2y = 2$ at (1,1).
5. Find equation of the line through $(\sqrt{32}, 0)$ and tangent to the circle with equation

   $x^2 + y^2 = 16$.
6. Suppose P(1,2) and Q(3, 0) are the endpoints of a diameter of a circle and L is the line tangent to the circle at Q.

   (a) Show that R(5, 2) is on L.

   (b) Find the area of ΔPQR, when R is the point given in (a).

4.3 Parabolas

By the end of this section, you should

- know the geometric definition of a parabola.
- know the meaning of vertex, focus, directrix, and axis of a parabola.
- be able to find equation of a parabola whose axis is horizontal or vertical.
- be able to identify equations representing parabolas.
- be able to find the vertex, focus, and directrix of a parabola and sketch the parabola.

4.3.1 Definition of a Parabola

Definition 4.3: Let L be a fixed line and F be a fixed point not on the line, both lying on the plane. A **parabola** is a set of points equidistant from L and F. The line L is called the **directrix** and the fixed point F is called the **focus** of the parabola.

This definition is illustrated by Figure 4.12.

- Note that the point halfway between the focus F and directrix L is on the parabola; it is called the **vertex,** denoted by V.
- |VF| is called the **focal length**.
- The line through F perpendicular to the directrix is called the **axis** of the parabola. It is the line of symmetry for the parabola.
- The chord BB′ through F perpendicular to the axis is called **latus rectum**.
- The length of the latus rectum, i.e, |BB′|, is called **focal width**.

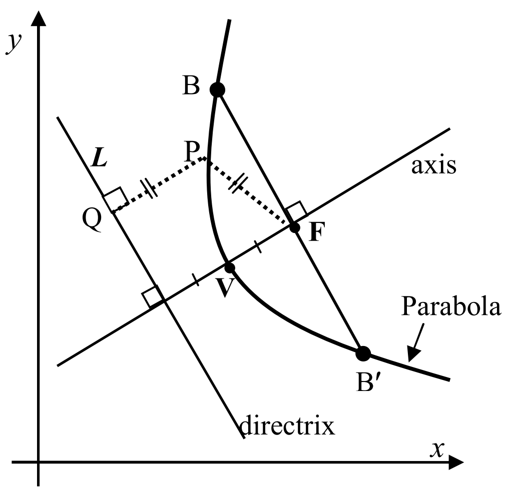


Figure 4.12: Parabola, $d(\mathrm{P},L) = |\mathrm{PF}|$

Letting $|\mathrm{VF}| = p$, you may show that $|\mathrm{BB}'| = 4p$; i.e., focal width is 4 times focal length.

If P(x,y) is any point on the parabola, then by the definition, the distance of P from the directrix is equal to the distance between P and the focus F. This is used to determine an equation of a parabola. To do this, we consider first the cases when the axis of the parabola is parallel to one of the coordinate axes.

Exercise 4.3.1

Use the definition of parabola and the given information to answer or solve each of the following problems.

1. Suppose the focal length of a parabola is p, for some $p > 0$. Then, show that the focal width (length of the latus rectum) of the parabola is $4p$.
2. Suppose the vertex of a parabola is the origin and its focus is F(0,1). Then,
   (a) What is the focal length of the parabola.
   (b) Find the equations of the axis and directrix of the parabola.
   (c) Find the endpoints of the latus rectum of the parabola.
   (d) Determine whether each of the following point is on the parabola or not.
   (i) (4, 4) (ii) (2, 2) (iii) (–4, 4) (iv) (4, –4) (v) (1, 1/4)

(Note: By the definition, a point is on the parabola iff its distances from the focus and from the directrix are equal.)

3. Suppose the vertex of a parabola is V(0, 1) and its directrix is the line $x = -2$. Then,
   (a) Find the equation of the axis of the parabola.
   (b) Find the focus of the parabola.
   (c) Find the length and endpoints of the latus rectum of the parabola.
   (d) Determine whether each of the following point is on the parabola or not.
   (i) (1, 0) (ii) (3, 0) (iii) (8, 9) (iv) (8, −7) (v) (8, 8)

4.3.2 Equation of Parabolas

I: Equation of a parabola whose axis is parallel to the *y*-axis:

A parabola whose axis is parallel to ***y***-axis is called **vertical parabola**. A vertical parabola is either open upward (as in Figure 4.13 (a)) or open downward (as in Figure 4.13 (b)).

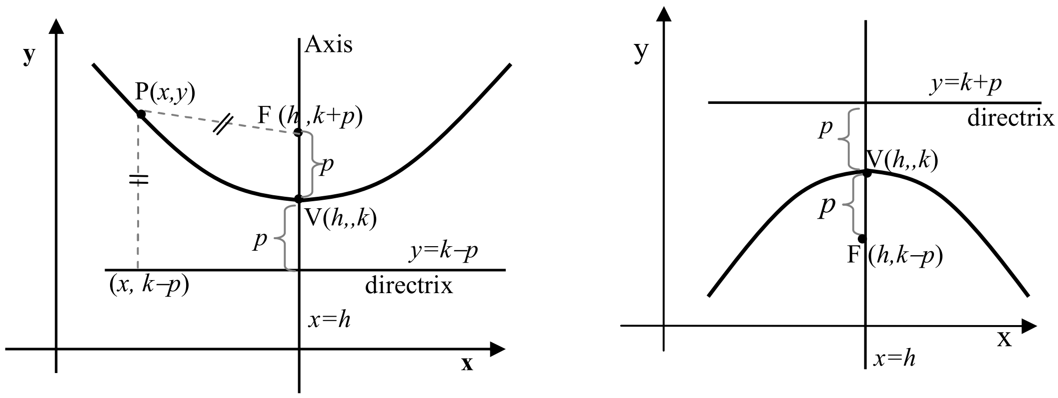


Figure 4.13: (a) parabola open upward (b) parabola open downward

Let p be the distance from vertex V(h,k) to the focus F of the parabola, i.e.,|VF| = p. Then, by the definition, F is located p units above V if the parabola opens upward and it is located p units below V if the parabola opens downward as indicated on Figure 4.13(a) and (b), respectively. To determine the desired equation, we first consider the case when the parabola opens upward.

Therefore, considering a vertical parabola with vertex V(h,k) that opens upward (Figure 4.13a), its focus is at F(h, $k+p$). ⇒ The equation of its directrix is $y = k-p$.

Then, for any point P(x,y) on the parabola, |PF| is equal to the distance between P and the directrix if and only if

$$\sqrt{(x-h)^2 + (y-k-p)^2} = y - k + p$$

Upon simplification, this becomes

$$(x-h)^2 = 4p(y-k)$$

called standard equation of a vertical parabola, vertex (h, k), focal length p, open upward.

In particular, if the vertex of a vertical parabola is at origin, i.e., $(h, k) = (0,0)$ and opens upward, then its equation is

$$x^2 = 4py$$ (In this case, its focus is at F(0, p), and its directrix is $y = -p$)

If a vertical parabola with vertex V(h, k) opens downward, then its directrix is above the parabola and its focus lies below the vertex (see Figure 4.13(b). In this case, the focus is at F(h, $k-p$), and its directrix is given by $y=k+p$. Moreover, following the same steps as above , the equation of this parabola becomes

$$(x-h)^2 = -4p(y-k)$$ (Standard equation of a vertical parabola, open downward, vertex (h, k) , and focal length p .)

In particular, if the vertex of a vertical parabola is at origin, i.e., $(h, k) = (0,0)$ and opens downward, then its equation is

$$x^2 = -4py$$ (In this case, its focus is at F(0,$-p$), and its directrix is $y = p$)

Example 4.14: Find the vertex, focal length, focus and directrix of the parabola $y = x^2$.

Solution: The given equation, $x^2 = y$, is the standard equation of the parabola with vertex at origin (0,0) and $4p = 1$ $\Rightarrow$ its focal length is $p = 1/4$. Since the parabola opens upward, its focus is p units above its vertex $\Rightarrow$ its focus is at F(0,1/4); and its directrix is horizontal line p units below its vertex $\Rightarrow$ its directrix is $y = -1/4$. You may sketch this parabola.

Example 4.15: If a parabola opens upward and the endpoints of its latus rectum are at $A(-4, 1)$ and $B(2, 1)$, then find the equation of the parabola, its directrix and sketch it.

Solution: Since the focus F of the parabola is at the midpoint of its latus rectum AB, we have $F = \left(\frac{-4+2}{2}, \frac{1+1}{2}\right) = (-1, 1)$, and focal width $4p = |AB| = 2 - (-4) = 6 \Rightarrow$ focal length $p = 3/2$.
Moreover, as the parabola opens upward its vertex is p units below its focus. That is,
V $(h, k) = (-1, 1-3/2) = (-1, -1/2)$. Therefore, the equation of the parabola is

$$(x+1)^2 = 6\left(y + \tfrac{1}{2}\right).$$

And its directrix is horizontal line p units below its vertex, which is $y = -1/2 - 3/2 = -2$.
The parabola is sketch in the Figure 4.14 .

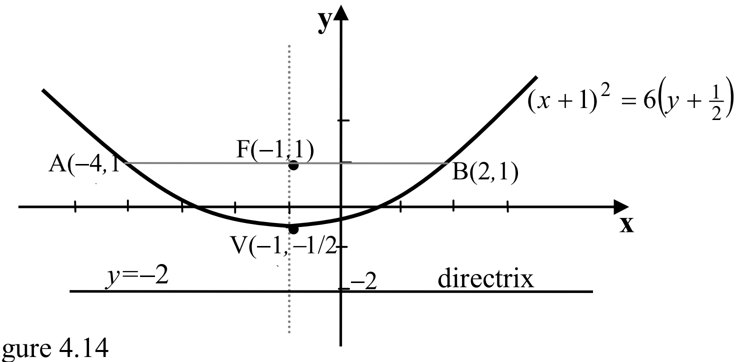


Figure 4.14

II: Equation of a parabola whose axis is parallel to the *x*-axis.

A parabola whose axis is parallel to *x*-axis is called **horizontal parabola**. Such parabola opens either to the right or to the left as shown in Figure 4.15 (a) and (b), respectively.

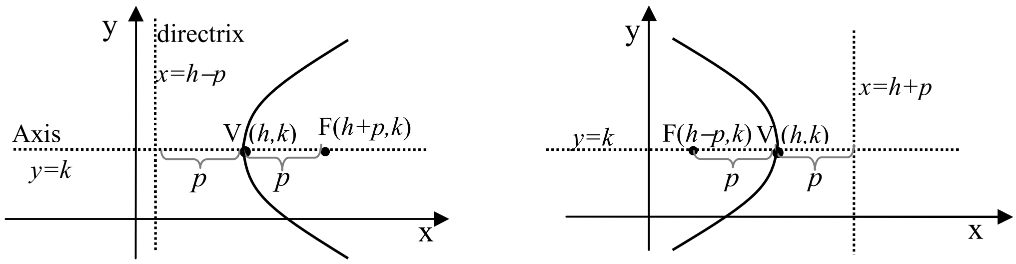


Figure 4.15: (a) Parabola open to the right (b) Parabola open to the left

The equations of such parabolas can be obtained by interchanging the role of x and y in the equations of the parabolas discussed previously. These equations are stated below. In both cases, let the vertex of the parabola be at $V(h,k)$.

- If a horizontal parabola **opens to the right (**as in Fig.4.15(a)), then its focus is to the right of V at $F(h+p, k)$, its directrix is $x = h - p$, and its equation is

$$(y-k)^2 = 4p(x-h)$$

- If a parabola **opens to the left (**as in Figure 4.15 (b)), then its focus is to the left of V at $F(h-p,k)$, its directrix is $x=h+p$, and its equation is:

$$(y-k)^2 = -4p(x-h)$$

If the vertices of these parabolas are at the origin (0,0), then you can obtain their corresponding equations by setting h=0 and k=0 in the above equations.

Example 4.16: Find the focus and directrix of the parabola $y^2+10x=0$ and sketch its graph.

Solution: The equation is $y^2=-10x$; and comparing this with the above equation, it is an equation of a parabola whose vertex is at (0,0), axis of symmetry is the x-axis, open to the left and $4p=10$, i.e., $p=5/2$. Thus, the focus is F=(–5/2,0) and its directrix is $x=5/2$. Its graph is sketched in Figure 4.16.

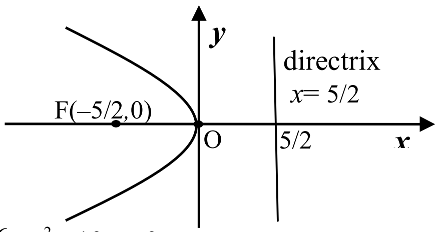


Figure 4.16: $y^2+10x=0$

Example 4.17: Find the focus and directrix of the parabola $y^2+4y+8x-4=0$ and sketch it.

Solution: The eaquation is $y^2+4y=-8x+4$. (Now complete the square of y-terms)

$\Rightarrow y^2+4y+\mathbf{2^2}=-8x+4+\mathbf{4}$

$\Rightarrow (y+2)^2=-8x+8$

$\Rightarrow (y+2)^2=-8(x-1)$

This is equation of a parabola with vertex at $(h, k)=(1,-2)$, open to the left and focal length p, where $4p=8 \Rightarrow p=2$. Therefore, its focus is

F= $(h-p, k)=(-1,-2)$, and directrix $x=h+p=3$. The parabola is sketched in Figure 4.17.

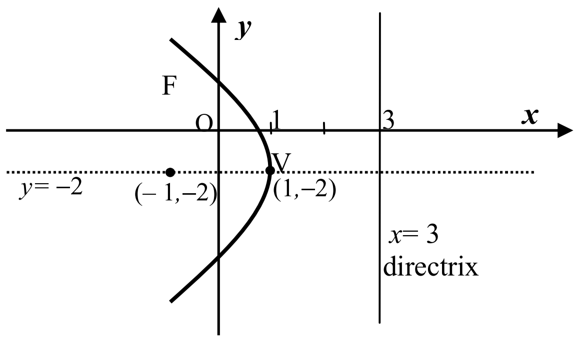


Figure 4.17: $y^2+4y+8x-4=0$

Remark:- An equation given as: $Ax^2+Dx+Ey+F=0$

or $Cy^2+Dx+Ey+F=0$

may represent a parabola whose axis is parallel to the ***y***-axis or parallel to the ***x***-axis, respectively. The vertex, focal length and focus for such parabolas can be identified after converting the equations into one of the standard forms by completing the square.

Exercise 4.3.2

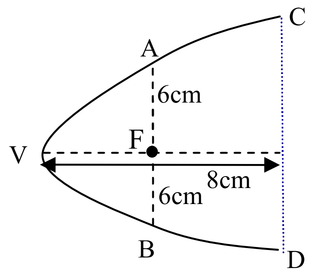
For questions 1 to 8, find an equation of the parabola with the given properties and sketch its graph.

1. Focus (0, 1) and directrix $y = -1$.
2. Focus (–1, 2) and directrix $y = -2$.
3. Focus (3/2, 0) and directrix $x = -3/2$.
4. Focus (–1,–2) and directrix $x = 0$.
5. Vertex (3, 2) and Focus (3, 3).
6. Vertex (5, –2) and Focus (–5, –2).
7. Vertex (1, 0) and directrix $x = -2$.
8. Vertex (0, 2) and directrix $y = 4$.

For questions 9 to 17 find the vertex, focus and directrix of the parabola and sketch it.

9. $y = 2x^2$
10. $8x^2 = -y$
11. $4x - y^2 = 0$
12. $x + y^2 = 0$
13. $x - 1 = (y+2)^2$
14. $(x+2)^2 = 8y - 24$
15. $y^2 + 8x + 6y + 25 = 0$
16. $y^2 - 2y - 4x + 9 = 0$
17. $-4x^2 + 4x - \frac{1}{2}y = 1$

18. Find an equation of the parabola that has a vertical axis, its vertex at (1, 0) and passing through (0,1).
19. The vertex and endpoints of the latus rectum of the parabola $x^2 = 36y$ forms a triangle. Find the area of the triangle.
20. P(4, 6) is a point on a parabola whose focus is at (0, 2) and directrix is parallel to x-axis.
    (a) Find an equation of the parabola, its vertex and directrix.
    (b) Determine the distance from P to the directrix.
21. An iron wire bent in the shape of a parabola has latus rectum of length 60cm. What is its focal length?
22. A cross-section of a parabolic reflector is shown in the figure below. A bulb is located at the focus and the opening at the focus, AB, is 12 cm. What is the diameter of the opening, CD, 8 cm from the vertex?



4.4 Ellipses

By the end of this section, you should

- know the geometric definition of an ellipse.
- know the meaning of the center, vertices, foci, major axis and minor axis of an ellipse.
- be able to find equation of an ellipse whose major axis is horizontal or vertical.
- be able to identify equations representing ellipses.
- be able to find the center, foci and vertices of an ellipse and sketch the ellipse.

4.4.1 Definition of an Ellipse

Definition 4.4: Let F and F′ be two fixed points in the plane. An **ellipse** is the locus or set of all points in the plane such that the sum of the distances from each point to F and F′ is constant. That is, a point P is on the ellipse if and only if $|PF| + |PF'| =$ constant. (See Figure 4.18).

The two fixed points, F and F′, are called **foci** (singular- **focus**) of the ellipse.

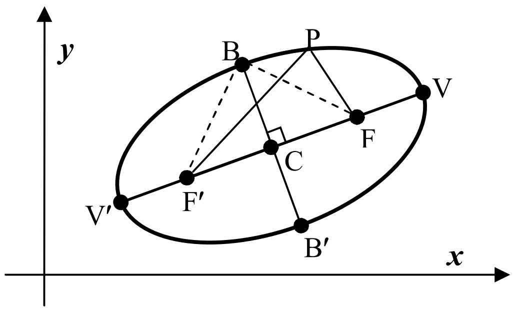


Figure 4.18: Ellipse: $|PF| + |PF'| =$ constant

Note also the following terminologies and relationships about ellipse.

- The midpoint C between the foci F′ and F is called the **center** of the ellipse.
- The longest diameter (longest chord) V′V through F′ and F is called the **major axis** of the ellipse; and the chord BB′ through C which is perpendicular to V′V is called **minor axis**.
- The endpoints of the major axis, V′ and V, are called the **vertices** of the ellipse.
- From the definition, $|V'F'| + |V'F| = |VF'| + |VF| \Rightarrow |V'F'| = |VF| \Rightarrow |CV'| = |CV|$. Hence, C is the midpoint of V′V. We denote the length of the major axis by $2a$. That is, $|CV| = a$.
  $\Rightarrow |VF'| + |VF| = |V'V| = 2a$.
  $\Rightarrow |PF'| + |PF| = 2a$, for any point P on the ellipse.
- We let $|BC| = b$. (You can show that C is the midpoint of BB′. So, $|B'C| = b$.)
- The distance from the center C to a focus F (or F′) is denoted by c, i.e., $|CF| = c = |CF'|$.

- Now, since |BF′|+ |BF| = $2a$ and BC is a perpendicular bisector of F′F, we obtain that |BF′| = |BF|= a . Hence, using the Pythagoras Theorem on ΔBCF, we obtain

$$b^2 + c^2 = a^2 \quad \text{or} \quad b^2 = a^2 - c^2.$$

(**Note:** $a \geq b$. If $a=b$, the ellipse would be a circle with radius $r = a = b$).

- The ratio of the distance between the two foci to the length of the major axis is called the **eccentricity** of the ellipse, and denoted by e. That is,

$$e = \frac{|F'F|}{|V'V|} = \frac{c}{a}. \qquad \text{(Note that } 0 < e < 1 \text{ because } 0 < c < a\text{)}$$

Exercise 4.4.1

Use the definition of ellipse and the given information to answer or solve each of the following problems.

1. Suppose F′ and F are the foci of an ellipse and B′ and B are the endpoints of the minor axis of the ellipse, as in Figure 4.18. Then, show that each of the followings hold.
   (a) ΔBF′F is isosceles triangle.
   (b) The quadrilateral BF′B′F is a rhombus.
   (c) FF′ is perpendicular bisector of BB′ ; and also BB′ is perpendicular bisector of FF′.
   (d) If the length of the major axis is $2a$, length of minor axis is |BB′| = $2b$, and |F′F|=$2c$, for some positive a, b, c, then
       (i) |BF| = a (ii) $a^2 = b^2 + c^2$
2. Suppose the vertices of an ellipse are (±2, 0) and its foci are (±1, 0).
   (a) Where is the center of the ellipse?
   (b) Find the endpoints of its minor axis.
   (c) Find the lengths of the major and minor axes.
   (d) Determine whether each of the following points is on the ellipse or not.
       (i) (1, 3/2) (ii) (3/2, –1) (iii) (–1, 3/2) (iv) (–1, –3/2) (v) (1, 1)
       (Note: By the definition, a point is on the ellipse iff the sum of its distances to the two foci is $2a$)
3. Suppose the endpoints of the major axis of an ellipse are (0, ±2) and the end points of its minor axis are (±1, 0).
   (a) Where is the center of the ellipse?
   (b) Find the coordinates of the foci.
   (c) Determine whether each of the following points is on the ellipse or not.
       (i) $\left(1/2, \sqrt{3}\right)$ (ii) $\left(\sqrt{2}, 1\right)$ (iii) $\left(-1/2, -\sqrt{3}\right)$ (iv) $\left(\sqrt{3}/2, 1\right)$
4. Suppose the endpoints of the minor axis of an ellipse are (1, ±3) and its eccentricity is 0.8. Find the coordinates of (a) the center, (b) the foci , (c) the vertices of the ellipse.

4.4.2 Equation of an Ellipse

In order to obtain the simplest equation for an ellipse, we place the ellipse at standard position. An ellipse is said to be at standard position when its center is at the origin and its major axis lies on either the ***x***-axis or **y**-axis.

I. Equation of an ellipse at standard position:

There are two possible situations, namely, when the major axis lies on x-axis (called horizontal ellipse) and when the major axis lies on y-axis (called vertical ellipse). We first consider a horizontal ellipse as in Figure 4.19

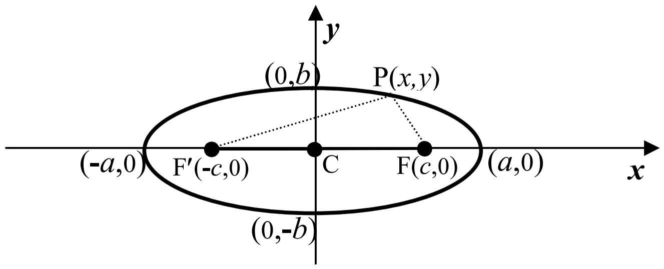


Figure 4.19: Horizontal ellipse at standard position

Let the center of the ellipse be at the origin, C(0,0) and foci at F′($-c$,0), F (c,0) and vertices at ($-a$,0) and (a,0) (see Figure 4.19). Then, a point P(x,y) is on the ellipse iff

$$|PF'| + |PF| = 2a\,.$$

That is, $\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} = 2a$

or $\sqrt{(x-c)^2+y^2} = 2a-\sqrt{(x+c)^2+y^2}$

Squaring both sides we get

$$x^2-2cx+c^2+y^2 = 4a^2-4a\sqrt{(x+c)^2+y^2}+x^2+2cx+c^2+y^2$$

which simplifies to $a\sqrt{(x+c)^2+y^2} = a^2+cx$

Again squaring both sides, we get $a^2(x^2+2cx+c^2+y^2) = a^4+2a^2cx+c^2x^2$

which becomes $(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$

Now recalling that $b^2 = a^2-c^2$ and dividing both sides by a^2b^2, the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

(Equation of horizontal ellipse at **standard position,** vertices ($\pm a$, 0), foci ($\pm c$, 0), where $c^2 = a^2 - b^2$)

For a vertical ellipse at standard position, the same procedure gives the equation

$$\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$$

(Equation of vertical ellipse at **standard position,** vertices (0, $\pm a$), foci (0, $\pm c$), where $c^2 = a^2 - b^2$)

Note: Notice that here, for vertical ellipse, the larger denominator a^2 is under y^2.

Example 4.18: Locate the vertices and foci of $16x^2 + 9y^2 = 144$ and sketch its graph.

Solution: Dividing both sides of the equation by 144, we get:

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \quad \text{or} \quad \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

This is equation of a vertical ellipse at standard position with a=4, so vertices at (0, ±4), and b=3; i.e., endpoints of the minor axis at (±3, 0). Since $c^2 = a^2 - b^2 = 7 \Rightarrow c = \sqrt{7}$, the foci are $(0, \pm\sqrt{7})$. The graph is sketched in Figure 4.20.

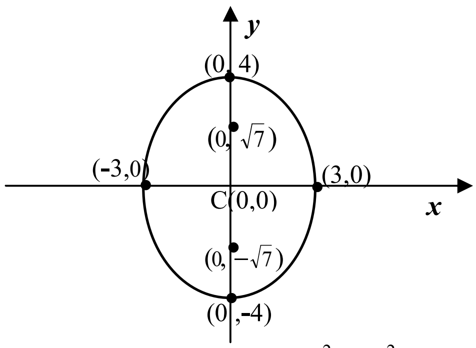


Figure 4.20: $16x^2 + 9y^2$

(II) Equation of shifted Ellipses:

When an ellipse is not at standard position but with center at a point C(h,k), then we can still obtain its equation by considering translation of the xy-axes in such a way that its origin translated to the point C(h,k). This result in a new $X'Y'$ coordinate system whose origin O′ is at C(h,k) so that the ellipse is at standard position relative to the $X'Y'$ system(see, Figure 4.21)

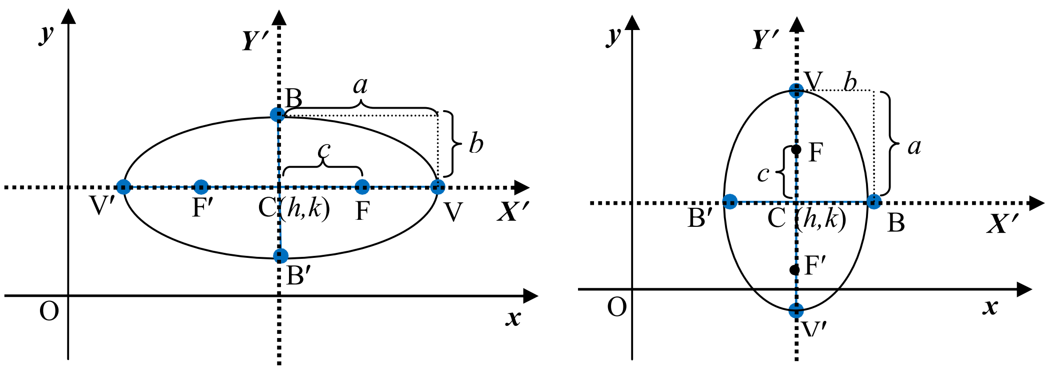


Fig. 4.21: (a) horizontal ellipse, center C(h,k) (b) vertical ellipse, center C(h,k)

Consequently, the equation of the horizontal and vertical ellipses relative to the new $X'Y'$ coordinate system with (x', y') coordinate points are

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \quad \text{and} \quad \frac{x'^2}{b^2} + \frac{y'^2}{a^2} = 1, \qquad . \ . \ . \ . \ . \ . \ . \ . \ . \ (I).$$

respectively. Since the origin of the new coordinate system is at the point (h,k) of the xy-coordinate system, the relationship between a point (x,y) of the $\boldsymbol{xy}$-coordinate system and (x',y') of the new coordinate system is given by $(x, y) = (x',y') + (h, k)$. That is,

$$x' = x - h, \quad \text{and} \quad y' = y - k.$$

Thus, in the original xy-coordinate system the equations of the horizontal and vertical ellipses with center C(h, k), lengths of major axis = $2a$ and minor axis = $2b$ are, respectively, given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

(Standard equation of horizontal ellipse with center C(h,k))

and

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

(Standard equation of vertical ellipse with center C(h,k))

Example 4.19: The endpoints of the major axis of an ellipse are at $(-3,4)$ and $(7,4)$ and its eccentricity is 0.6. Find the equation of the ellipse and its foci.

Solution: The given vertices are at V′$(-3,4)$ and V$(7,4)$ implies that $2a = |V'V| = 10 \Rightarrow a = 5$; and the center C$(h,k)$ is the midpoint of V′V $\Rightarrow (h,k) = (\frac{-3+7}{2}, \frac{4+4}{2}) = (2,4)$. Moreover, eccentricity = $c/a = 0.6 \Rightarrow c = 5\times 0.6 = 3$. Hence, $b^2 = a^2 - c^2 = 25 - 9 = 16$. Note that the major axis V'V is horizontal. Therefore, using the standard equation of a horizontal ellipse, the equation of the ellipse is

$$\frac{(x-2)^2}{25} + \frac{(y-4)^2}{16} = 1.$$

Now, as the center $(h,k) = (2,4)$, $c=3$ and V′V is horizontal, the foci are at $(h \pm c, k) = (2 \pm 3, 4)$. That is, the foci are at F′$(-1, 4)$ and F$(5, 4)$. Moreover, the endpoints of major axis are at $(h, k \pm b) = (2, 4 \pm 4) \Rightarrow$ B′=(2,0) and B=(2,8). The graph of the ellipse is sketched in Figure 4.22.

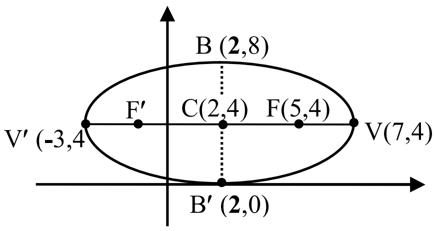


Figure 4.22

Example 4.20: Find the center, foci and vertices of $4x^2 + y^2 + 8x = 0$ and sketch its graph

Solution: Group the x-terms of the equation and complete the square:

$$4(x^2 + 2x) + y^2 = 0$$

$$\Rightarrow 4(x^2+2x+1) + y^2 = 4 \quad \text{(divide both sides by 4)}$$

$$\Rightarrow (x+1)^2 + \frac{y^2}{4} = 1$$

This is equation of a vertical ellipse (major axis parallel to the y-axis), center C=$(h,k) = (-1,0)$,

$a = 2,\ b = 1. \Rightarrow c^2 = a^2 - b^2 = 4 - 1 \quad \Rightarrow c = \sqrt{3}$

Thus, foci : F′$(-1, -\sqrt{3})$ and F$(-1, \sqrt{3})$,

Vertices: V = (–1, 2), V′= (–1, –2);

Endpoints of minor axis: B=(0,0), B′=(–2,0);

The graph of the ellipse is sketched in Figure 4.23.

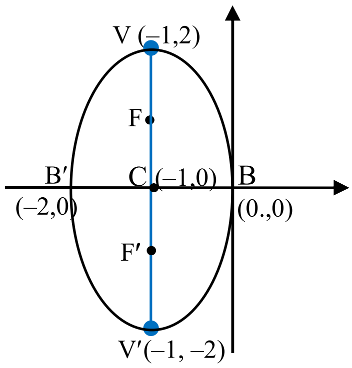


Figure 4.23: $4x^2 + y^2 + 8x = 0$

Remark: Consider the equation: $Ax^2 + Cy^2 + Dx + Ey + F = 0$,

when A and C have the same sign. So, without lose of generality, let A >0 and C >0.

By completing the squares you can show that this equation is equivalent to

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{D^2C + E^2A - 4ACF}{4AC}.$$

From this you can conclude that the given equation represents:-

- an ellipse with center $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$ if $D^2C + E^2A - 4ACF > 0$.
- If $D^2C + E^2A - 4ACF = 0$, the equation is satisfied by the point $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$ only. In this case, the locus of the equation is called a **point-ellipse** (degenerate ellipse).
- If $D^2C + E^2A - 4ACF < 0$, then the equation has no locus.

Exercise 4.4.2

For questions 1 to 13, find an equation of the ellipse with the given properties and sketch its graph.

1. Foci at $(\pm 2, 0)$ and a vertex at (5, 0)
2. A focus at (0, –3) and vertices at $(0, \pm 5)$
3. Foci at (2, 3) , (2,7) and a vertex at (2, 0)
4. Foci at (0, –1) , (8,–1) and a vertex at (9, –1)
5. Center at (6,1) , one focus at (3,1) and one vertex at (10, 1)
6. Foci at $(2, \pm 1)$ and the length of the major axis is 4.
7. Foci at (2, 0), (2, 6) and the length of the minor axis is 5.

8. The distance between its foci is $2\sqrt{5}$ and the endpoints of its minor axis are (–1, –2) and (3,–2).
9. Vertices at (±5, 0) and the ellipse passes through (−3, 4).
10. Center at (1, 4), a vertex at (10, 4) , and one of the endpoints of the minor axis is (1, 2).
11. The ellipse passes through (−1, 1) and $\left(\frac{1}{2},-2\right)$ with center at origin.
12. The endpoints of the major axis are (3, −4) and (3, 4) , and the ellipse passes through the origin
13. The endpoints of the minor axis are (3, −2) and (3, 2) , and the ellipse passes through the origin

For questions 14 to 22 find the center, foci and vertices of the ellipse having the given equation and sketch its graph.

14. $\frac{x^2}{9}+\frac{y^2}{5}=1$
15. $5x^2+y^2=25$
16. $x^2+9y^2=9$
17. $\frac{(x-2)^2}{9}+\frac{(y+3)^2}{16}=1$
18. $(x+1)^2+2(y+2)^2=3$
19. $x^2+9y^2-2x+18y+1=0$
20. $9x^2+4y^2-18x=27$
21. $x^2+2y^2-6x+4y=-7$
22. $4x^2+y^2+2x-10y=6$
23. Consider the equation $2x^2+4y^2+8x-16y+F=0$. Find all values of F such that the graph of the equation
    (a) is an ellipse. (b) is a point. (c) consists of no points at all.

4.5 Hyperbolas

By the end of this section, you should

- know the geometric definition of a hyperbola.
- know the meaning of the center, vertices, foci and transverse axis of a hyperbola.
- be able to find equation of a hyperbola whose transverse axis is horizontal or vertical.
- be able to identify equations representing hyperbolas.
- be able to find the center, vertices, foci, and asymptotes of a hyperbola and sketch the hyperbola.

4.5.1 Definition of a hyperbola

Definition 4.5: Let F and F′ be two fixed points in the plane. A **hyperbola** is the set of all points in the plane such that the difference of the distance of each point from F and F′ is constant. We shall denote the constant by $2a$, for some $a > 0$. That is, a point P is on the hyperbola if and only if $|PF'| - |PF| = 2a$ (or $|PF| - |PF'| = 2a$, whichever is positive) . The two fixed points F and F′ are called the foci of the hyperbola.

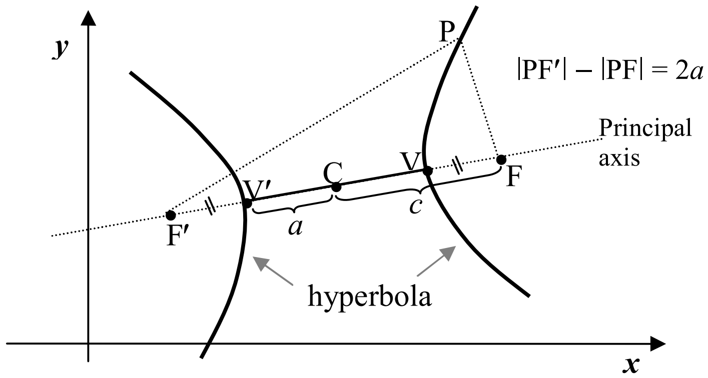


Figure 4.24: Hyperbola

Figure 4.24 illustrates the definition of hyperbola. Notice that the definition of hyperbola is similar to that of an ellipse, the only change is that the sum of distances has become the difference of distances. Here, for the difference of any two unequal values, we take the higher value minus the smaller so that $a > 0$ in the definition. The following terminologies, notations and relationships are also important with regard to a hyperbola. Refer to Figure 4.24 for the following discussion.

- The line through the two foci F′ and F is called the **principal axis** of the hyperbola. The point on the principal axis at halfway between the two foci, that is, the midpoint of F′F, is called the **center** of the hyperbola and represented by C. We denote the distance between the two foci by $2c$. That is, $|F'F| = 2c$ or $|CF| = c = |CF'|$. Noting also that $|PF'| < |F'F| + |PF|$ in $\Delta\, PF'F$ and $|PF'| - |PF| = 2a$, you can show that $a < c$.
- The points V′ and V where the hyperbola crosses the principal axis are called **vertices** of the hyperbola. The line segment V′V is called the **transverse axis** of the hyperbola. So, as V′ and V are on the hyperbola, the definition requires that $|V'F| - |V'F'| = |VF'| - |VF|$. From this, you can obtain that $|V'F'| = |VF|$. Consequently,
  (i) C is the midpoint of also V′V; that is, $|CV'| = |CV|$ **.**
  (ii) $|V'V| = |V'F| - |VF| = |V'F| - |V'F'| = 2a$. (The length of the transverse axis is $2a$)
  (iii) $|V'C| = a = |CV|$ (This follows from (i) and (ii).)

- The **eccentricity** ***e*** of a hyperbola is defined to be the ratio of the distance between its foci to the length of its transverse axis. That is, similar to the definition of eccentricity of an ellipse, the eccentricity of a hyperbola is

$$e = \frac{|F'F|}{|V'V|} = \frac{c}{a} \qquad \text{(But here, } e > 1 \text{ because } c > a\text{)}$$

Exercise 4.5.1

Use the definition of hyperbola and the given information to answer or solve each of the following problems.

1. Suppose C is the center, F′ and F are the foci, and V′ and V are the vertices of the hyperbola, as in Figure 4.24, with $|CV| = a$ and $|CF| = c$. Then, show that each of the followings hold.
   (a) If P is any point on the hyperbola, then $|PF| - |PF'| = \pm 2a$.
   (Note: Taking that $|PF| - |PF'| = k$, a constant, show that $k = \pm 2a$.)
   (b) $a > c$.
2. Consider a hyperbola whose foci are $(\pm 2, 0)$ and contains the point P(2, 3).
   (a) Where is the center of the hyperbola?
   (b) Determine the principal axis of the hyperbola.
   (c) Find the length of the transverse axis of the hyperbola.
   (d) Find the coordinates of the vertices of the hyperbola.
   (e) Determine whether each of the following points is on the hyperbola or not.
   (i) $(-2, 3)$ (ii) $(-2, -3)$ (iii) $(2, -3)$ (iv) $(3, 4)$ (v) $(\sqrt{13}, 6)$
3. Suppose the vertices of a hyperbola are at $(0, \pm 2)$ and its eccentricity is 1.5. Then,
   (a) Find the foci of the hyperbola.
   (b)) Determine whether each of the following points is on the hyperbola or not.
   (i) $(\sqrt{5}, 3)$ (ii) $(2, 3)$ (iii) $(\sqrt{5}, -3)$ (iv) $(3, \sqrt{5})$

4.5.2 Equation of a hyperbola

We are now ready to derive equation of a hyperbola. But, for simplicity, we consider first the equation of a standard hyperbola with center at origin. A standard hyperbola is the one whose principal axis (or transverse axis) is parallel to either of the coordinate axes.

I. Equation of a standard hyperbola with center at origin.

There are two possible situations, namely, when the transverse axis lies on x-axis (called horizontal hyperbola) and when the transverse axis lies on y-axis (called vertical hyperbola). We first consider a horizontal hyperbola with center C(0,0), vertices V′($-a$, 0), V (a, 0) and foci F′($-c$, 0), F (c, 0).

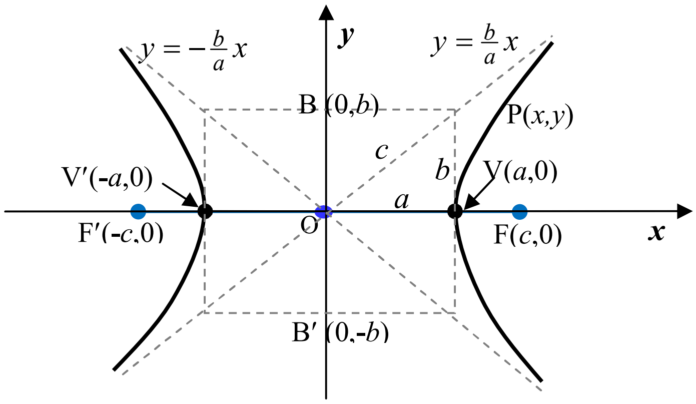


Figure 4.25: Horizontal hyperbola centered at origin

Notice that $c^2 - a^2 > 0$ as $c > a$. Hence, we can put $b^2 = c^2 - a^2$ for some positive b. That is, $a^2 + b^2 = c^2$ so that a, b, c are sides of a right triangle (see, Figure 4.25). The line segment BB′ perpendicular to the transverse axis at C and with endpoints B(0,b) and B′(0,$-b$) is called **conjugate axis** of the hyperbola. Observe that the midpoint of the conjugate axis is C and its length is $|BB'| = 2b$. (b will play important role in equation of the hyperbola and its graph).

Now, for any point P(x,y) on the hyperbola it holds that $|PF'| - |PF| = 2a$.

That is, $\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$

or $\sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$

Squaring both sides we get

$$x^2 + 2cx + c^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

which simplifies to $a\sqrt{(x-c)^2 + y^2} = cx - a^2$

Again squaring both sides and rearranging, we get $(c^2 - a^2)x^2 - y^2 = a^2(c^2 - a^2 + y^2)$.

Recall that we set $b^2 = c^2 - a^2$. So, using this in the above equation and dividing both sides by a^2b^2, the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(Equation of horizontal hyperbola with center C(0,0), vertices $(\pm a, 0)$, foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$)

Note that this hyperbola has no y-intercept because if $x = 0$, then $-y^2 = b^2$ which is not possible. The hyperbola is symmetric with respect to both x-and y- axes.

Also, from this equation we get

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1 \quad \text{implies that} \quad x^2 \geq a^2 \text{ . So, } |x| = \sqrt{x^2} \geq \sqrt{a^2} = a.$$

Therefore, we have $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its *branches*. Moreover, if we solve for y from the equation we get $y = \pm\frac{b}{a}\sqrt{x^2 - a^2} \to \pm\frac{b}{a}x$ as $x \to \infty$.

This means the hyperbola will approach (but never reaches) the line $y = \pm\frac{b}{a}x$ as x gets larger and larger. That is, the lines $y = \pm\frac{b}{a}x$ are the **asymptotes** of the hyperbola.

In sketching a hyperbola, it is best to draw the rectangle formed by the line $y = \pm b$ and $x = \pm a$ and the to draw the asymptotes which are along the diagonals of the rectangle (as shown by the dashed lines in Figure 4.25). The hyperbola lies outside the rectangle and inside the asymptotes. It opens around the foci.

Example 4.21: Find an equation of the hyperbola whose foci are F′(–5, 0) and F(5, 0) and contains point P(5, 16/3).

Solution: It is horizontal hyperbola with center (0,0) and $c = 5$. In addition, as P(5, 16/3) is on the hyperbola we have that |PF′| – |PF| = 2a**.** That is,

$$\sqrt{(5+5)^2 + \left(\tfrac{16}{3}\right)^2} - \sqrt{(5-5)^2 + \left(\tfrac{16}{3}\right)^2} = 2a$$

$\Rightarrow$ a =3. (So, its vertices are (–3, 0) and (3, 0)).

Now, using the relationship $b^2 = c^2 - a^2$, we get $b^2 = 25 - 9 = 16$.

Therefore, the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

You may find the asymptotes and sketch the hyperbola.

For a vertical hyperbola with center at origin (i.e., when transverse axis lies on y-axis), by reversing the role of x and y we obtain the following equation which is illustrated in Figure 4.26.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

(Equation of vertical hyperbola with center C(0,0), foci $(0, \pm c)$, vertices $(0, \pm a)$, where $c^2 = a^2 + b^2$ and asymptotes $y = \pm(a/b)x$)

Note:
- For a vertical hyperbola, the coefficient of y^2 is positive and that of x^2 is negative .
- a^2 is always the denominator of the positive term.

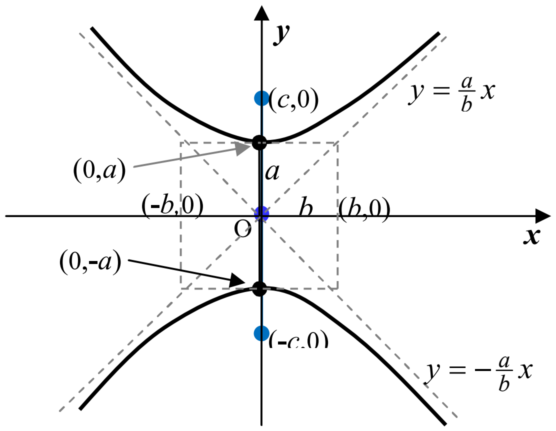


Figure 4.26: Vertical hyperbola centered at origin

Example 4.22: Find the foci and equation of the hyperbola with vertices V′(0,–1) and V(0, 1) and an asymptote $y = 2x$.

Solution: It is a vertical hyperbola with center C(0,0) and $a = |CV| = 1$. Since an asymptote of such vertical hyperbola is $y = (a/b)x$ and the slope of the given asymptote is 2, we have $a/b = 2$ $\Rightarrow$ $1/b = 2 \Rightarrow b = 1/2$. Thus, $c^2 = a^2 + b^2 = 1 + 1/4 = 5/4$.

So, the foci are $\left(0, \pm\sqrt{5}/2\right)$ and the equation of the hyperbola is $y^2 - 4x^2 = 1$.

(You may sketch the hyperbola)

(II) Equation of shifted hyperbolas:

The center of a horizontal or vertical hyperbola may be not at origin but at some other point C(h,k) as shown in Figure 4.27. In this case, we form the equation of the hyperbolas by using the translation of the xy-coordinate system that shifts its origin to the point C(h, k). As discussed in Section 4.4, the effect of this translation is just replacing x and y by $x–h$ and $y–k$, respectively, in the equation of the desired hyperbola.

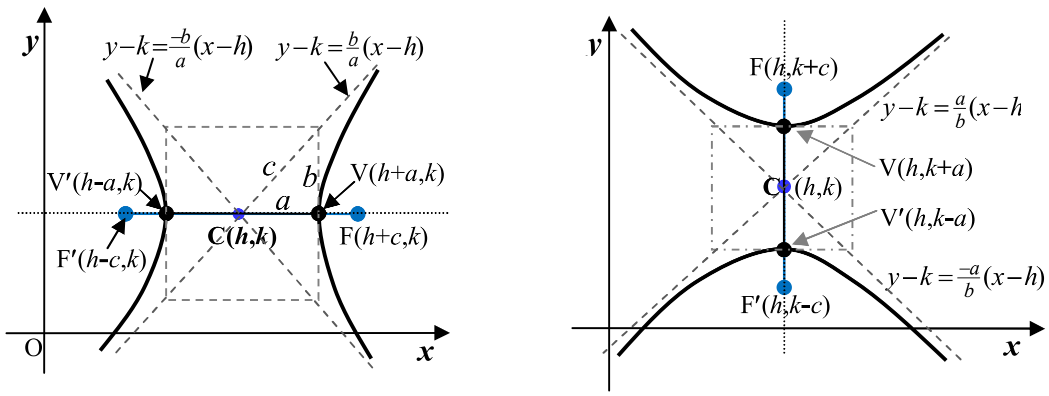


Figure 4.27: (a) Horizontal hyperbola with center C(h,k)

(b) Vertical hyperbola with center C(h,k)

Therefore, the standard equation of a horizontal hyperbola (transverse axis parallel to x-axis) with center C(h,k), length of transverse axis $=2a$, and length of conjugate axis $=2b$ is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Center: C(h,k),

Vertices: V′ ($h–a,k$), V($h+a$, k),

Foci : F′($h–c$, k), F($h+c$, k), where $c^2 = a^2 + b^2$

Asymptotes: $y-k=\pm\frac{b}{a}(x-h)$

Similarly, the standard equation of a vertical hyperbola (transverse axis parallel to y-axis) with center C(h,k), length of transverse axis $=2a$, and length of conjugate axis $=2b$ is

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Center: C(h,k),

Vertices: V′ ($h,k–a$), V(h, $k+a$),

Foci : F′(h, $k–c$), F(h, $k+c$), where $c^2 = a^2 + b^2$

Asymptotes: $y-k=\pm\frac{a}{b}(x-h)$

Example 4.23: Find the foci, vertices and the asymptotes of the hyperbola whose equation is

$$4(x+1)^2 - (y-2)^2 = 4$$

and sketch the hyperbola.

Solution: Dividing both sides of the equation by 4 yields

$$(x+1)^2 - \frac{(y-2)^2}{4} = 1.$$

This is equation of a hyperbola with center C(–1, 2). Note that the 'x^2-term' is positive indicates that the hyperbola is horizontal (principal axis y=2), a=1, b=2, and $c^2= a^2+b^2 \Rightarrow c=\sqrt{5}$. As a result the foci are at $\left(-1-\sqrt{5},2\right)$ and $\left(-1+\sqrt{5},2\right)$, vertices are at (–2,2) and (0,2) and the asymptotes are the lines y–2=±2(x+1), that is, y=2x+4 and y=–2x. Consequently, the hyperbola is sketched as in Figure 4.28.

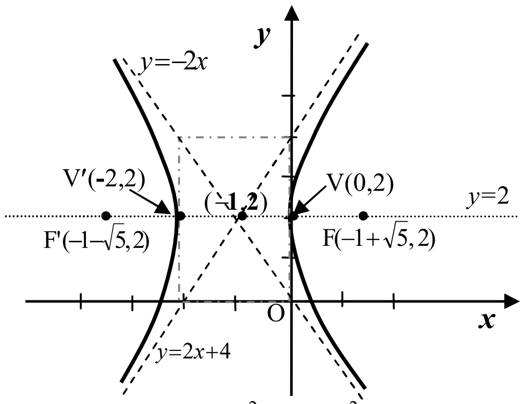


Figure 4.28: $4(x+1)^2 - (y-2)^2 = 4$

Example 4.24: Find the foci of the conic $9x^2-4y^2-72x+8y+176=0$ and sketch its graph.

Solution: Group the x-terms and y-terms of the equation and complete their squares:

$\Rightarrow\ 9x^2-72x-4y^2+8y=-176$ (Multiply both sides by –1)

$\Rightarrow\ -9x^2+72x+4y^2-8y=176$

$\Rightarrow\ 4(y^2-2y)-\ 9(x^2-8x)=176$

$\Rightarrow\ 4(y^2-2y+\mathbf{1^2})-\ 9(x^2-8x+\mathbf{4^2})=176+\mathbf{4}-\mathbf{144}$

$\Rightarrow\ 4(y-1)^2-\ 9(x-4)^2=36$ (Next, divide each by 36)

$\Rightarrow\ \dfrac{(y-1)^2}{9}-\dfrac{(x-4)^2}{4}=1$

This is standard equation of a hyperbola whose transverse axis is parallel to the y-axis (as its 'y^2 term' is positive) with center C (4,1), a^2=9 and b^2=4 . $\Rightarrow c^2=a^2+b^2=13 \Rightarrow c=\sqrt{13}$.
Thus, foci are $F'(4,1-\sqrt{13})$ and $F(4,1+\sqrt{13})$, and vertices (4, 1±3), i.e,, V′(4,–2) and V(4, 4). Moreover, the asymptotes are $y-k=\pm\frac{a}{b}(x-h)$. Hence, the asymptotes are

$\boldsymbol{l}_1$: $y-1=\frac{3}{2}(x-4)$ and $\boldsymbol{l}_2$: $y-1=-\frac{3}{2}(x-4)$. The hyperbola is sketched in Figure 4.29

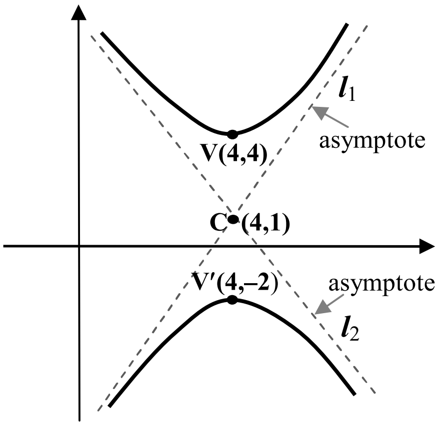


Figure 4.29: $\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$

Example 4.25: Determine the locus or type of the conic section given by the equation

$$-x^2 + y^2 + 4x - 2y = 3.$$

Solution: Grouping the *x*-terms and *y*-terms of the equation and completing their squares yield

$$(y-1)^2 - (x-2)^2 = 0$$

$$\Rightarrow \quad (y-1)^2 = (x-2)^2$$

$$\Rightarrow \quad y-1 = \pm\sqrt{(x-2)^2} = \pm(x-2)$$

This represents pair of two lines intersecting at (2, 1), namely, $y = x-1$ and $y = -x+3$.

Remark: Consider the equation: $Ax^2 + Cy^2 + Dx + Ey + F = 0$ when $AC < 0$; (i.e., A and C have opposite signs). Then, by completing the squares of x-terms and y-terms you can convert the equation to the following form:

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{D^2C + E^2A - 4ACF}{4AC}.$$

Now, letting $\Delta = D^2C + E^2A - 4ACF$, you can conclude the following:

- If $\Delta \neq 0$, the equation represents a hyperbola with center $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.
- If $\Delta = 0$, the equation becomes $y + \frac{E}{2C} = \pm\sqrt{\left|\frac{A}{C}\right|}\left(x + \frac{D}{2A}\right)$ which are two lines intersecting at $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$. In this case, it is called **degenerate** hyperbola.

Exercise 4.5.2

For questions 1 to 9, find an equation of the hyperbola having the given properties and sketch its graph.

1. Center at the origin, a focus at (5, 0), and a vertex at (3, 0)
2. Center at the origin, a focus at (0,–5), and a vertex at (0, –3).
3. Center at the origin, ***x***-intercepts ±3, an asymptote $y = 2x$.
4. Center at the origin, a vertex at (2, 0) , and passing through $(4, \sqrt{3})$.
5. Center at (4, 2), a vertex at (7, 2) , and an asymptote $3y = 4x - 10$.
6. Foci at (–2, –1) and $F_2(-2, 9)$, length of transverse axis 6.
7. Foci at (1, 3) and (7, 3), and vertices at (2,3) and (6, 3).
8. Vertices at (±3, 0), and asymptotes $y = \pm 2x$
9. Eccentricity e=1.5, endpoints of transversal axis at $(2, 2)$ and $(6, 2)$.

For questions 10 to 17 find the center, foci, vertices and asymptotes of the hyperbola having the given equation and sketch its graph.

10. $\frac{x^2}{64} - \frac{y^2}{36} = 1$
11. $y^2 - x^2 = 9$
12. $x^2 - y^2 = 9$
13. $(y+1)^2 - 4(y+2)^2 = 8$
14. $\frac{(x-2)^2}{9} - \frac{(y+3)^2}{16} = 1$
15. $4x^2 - y^2 + 2y - 5 = 0$
16. $2x^2 - 3y^2 - 4x + 12y + 8 = 0$
17. $-16x^2 + 9y^2 - 64x + 90y + 305 = 0$
18. Find an equation of hyperbola whose major axis is parallel to the x-axis, has a focus at $(2,1)$ and its vertices are at the endpoints of a diameter of the circle $x^2 + y^2 - 2y = 0$.
19. A satellite moves along a hyperbolic curve whose horizontal transverse axis is 24 km and an asymptote $y = \frac{5}{12}x + 2$. Then what is the eccentricity of the hyperbola?

20 Two regions A and B are separated by a sea. The shores are roughly in a shape of hyperbolic curves with asymptotes $y = \pm 3x$ and a focus at (30,0) taking a coordinate system with origin at the center of the hyperbola. What is the shortest distance between the regions in kms?

21. Determine the type of curve represented by the equation

$$\frac{x^2}{k} + \frac{y^2}{k-16} = 1$$

In each of the following cases: (a) $k < 0$, (b) $0 < k < 16$, (c) $k > 16$

4.6 The General Second Degree Equation

By the end of this section, you should

- know the general form of second degree equation representing conic sections whose lines of symmetry are not necessarily parallel to the coordinate axes.
- know the rotation formula for rotating the coordinate axes.
- be able to find equivalent equation of a conic section under rotation of the reference axes.
- be able to apply the rotation formula to find a suitable coordinate system in which a given general second degree equation is converted to a simpler standard form.
- be able to convert a given general second degree equation to an equivalent simpler standard form of equation of a conic section.
- be able to identify a conic section that a given general second degree equation represents and sketch the corresponding conic section.

In the previous sections we have seen that, except in degenerate cases, the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is a circle, parabola, ellipse or hyperbola. The construction of these equations was based on the assumption that the axis of symmetry of a conic section is parallel to one of the coordinate axes. The assumption seems to be quite restrictive because the axis of symmetry for a parabola, ellipse, or hyperbola can be any oblique line as indicated in their corresponding definitions (See Figures 4.12, 4.18 and 4.24).

However, the reason why we have assumed that is not only for simplicity but there is always a coordinate system whose one of the axes is parallel to a desired line of symmetry. In particular, we can rotate the axes of our *xy*-coordinate system, whenever needed, so as to form a new *x'y'*-coordinate system such that either the x'- axis or y'-axis is parallel to the desired line of symmetry. Toward this end, let us review the notion of rotation of axes.

4.6.1 Rotation of Coordinate Axes

A rotation of the x and y coordinate axes by an angle θ about the origin O(0,0) creates a new $x'y'$-coordinate system whose x'-axis is the line obtained by rotating the x-axis by angle θ about O and y'-axis is the line obtained by rotating the y-axis in the same way. This makes a point P to have two sets of coordinates denoted by (x,y) and (x',y') relative to the xy- and $x'y'$-coordinate axes, respectively.(See Figure 4.30).

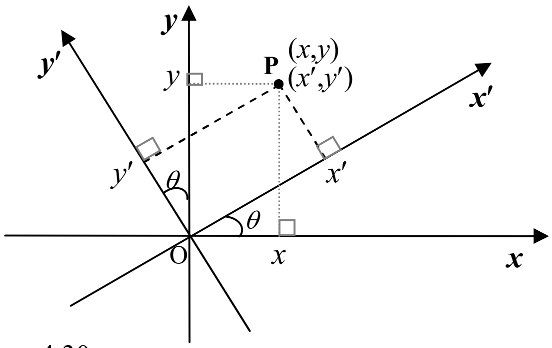


Figure: 4.30

The angle θ considered in the above discussion is called the angle of rotation. Our aim is to find the relationships between the coordinates (x,y) and the coordinates (x',y') of the same point P.

To find this relationships, let P(x,y) be any point in xy-plane, θ be an angle of rotation (i.e., θ is angle between x and x' axes) and ϕ be the angle between OP and x'-axis (See Figure 4.31).
So, letting $|OP| = r$ observe that

$$x' = r\cos\phi, \qquad y' = r\sin\phi \quad . \ . \ . \ . \ . \ . \ . \quad (1)$$

and

$$x = r\cos(\theta+\phi), \qquad y = r\sin(\theta+\phi) \quad . \ . \ . \ . \quad (2)$$

Then, using the trigonometric identities

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and (1), the equations in (2) become

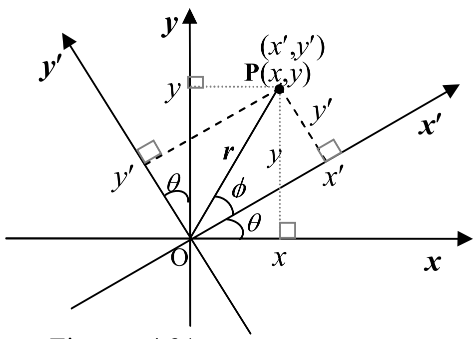


Figure: 4.31

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \quad . \ . \ . \quad (3)$$

Moreover, these equations can be solved for x' and y' in terms of x and y to obtain

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \quad . \ . \ . \quad (4)$$

The Equations (3) and (4) are called **rotation formulas**. It follows that if the angle of rotation θ is given, then Equation (3) can be used to determine the x and y coordinates of a point P if we know its x' and y' coordinates. Similarly, Equation (4) can be used to determine the x' and y' coordinates of P if we know its x and y coordinates.

Example 4.26: Suppose the $\boldsymbol{x}$ and $\boldsymbol{y}$ coordinate axes are rotated by π/4 about the origin.

(a) Find the coordinates of P(1, 2) relative to the new $\boldsymbol{x}'$ and $\boldsymbol{y}'$ axes.

(b) Find the equation of the curve $xy = 1$ relative to the new $x'y'$-coordinate system and sketch its graph.

Solution: The given information about P and the curve are relative to the xy-coordinate system and we need to express them in terms of x' and y' coordinates relative to the new $x'y'$-coordinate system obtained under the rotation of the original axes by $\theta = \pi/4$ rad about the origin. Thus, we use $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ in the relevant rotation formula to obtain the following.

(a) Since P(1,2) has the coordinates x=1 and y=2, its x' and y' coordinates are, using formula (4)

$$x' = \tfrac{\sqrt{2}}{2}(1) + \tfrac{\sqrt{2}}{2}(2) = \tfrac{3\sqrt{2}}{2}$$
$$y' = -\tfrac{\sqrt{2}}{2}(1) + \tfrac{\sqrt{2}}{2}(2) = \tfrac{\sqrt{2}}{2}$$

Therefore, the coordinates of P relative to the new $\boldsymbol{x}'$ and $\boldsymbol{y}'$ axes are $\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

(b) We need to express x and y in the equation xy =1 in terms of x' and y' using the rotation formula (3). So, again since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, we obtain from formula (3):

$$x = \tfrac{\sqrt{2}}{2}x' - \tfrac{\sqrt{2}}{2}y' \quad \text{and} \quad y = \tfrac{\sqrt{2}}{2}x' + \tfrac{\sqrt{2}}{2}y'$$

Therefore, $xy = 1 \Rightarrow \left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) = 1$

$$\Rightarrow \left(\tfrac{\sqrt{2}}{2}x'\right)^2 - \left(\tfrac{\sqrt{2}}{2}y'\right)^2 = 1$$

$$\Rightarrow \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

Note that this is an equation of a hyperbola with center at origin vertices $(-\sqrt{2},0)$ and $(\sqrt{2},0)$ in the $x'y'$-coordinate system with principal axis on $\boldsymbol{x}'$-axis. Since the $\boldsymbol{x}$ and **y**- axes were rotated though an angle of π/4 to obtain $\boldsymbol{x}'$ and $\boldsymbol{y}'$-axes, the hyperbola can be sketched as in Figure 4.32. (You may use Formula (3) to show that the vertices $(-\sqrt{2},0)$ and $(\sqrt{2},0)$ are $(-1,-1)$ and $(1,1)$, respectively, relative to the $\boldsymbol{x}$ and $\boldsymbol{y}$-axes).

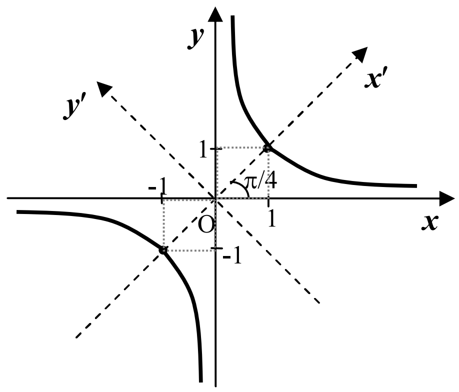


Figure 4.32: $xy = 1$

Example 4.27: Find an equation of the ellipse whose center is the origin, vertices are (−4,−3) and (4, 3), and length of minor axis is 6.

Solution: The position of the ellipse is as shown in Figure 4.33.

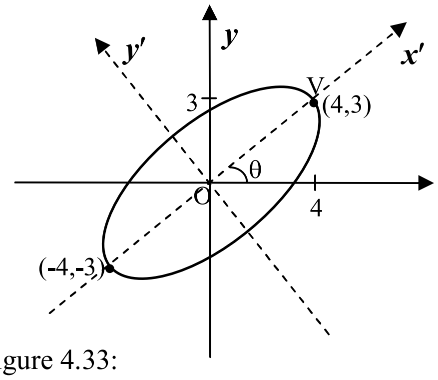


Figure 4.33:

To apply the standard equation of ellipse we use the $x'y'$-coordinate system such that the $\boldsymbol{x'}$-axis coincide with the major axis of the ellipse. Therefore, the equation of the ellipse relative to the $x'y'$ system is

$$\frac{x'^2}{a^2} + \frac{y'^2}{b'^2} = 1.$$

Moreover, from the given information, $a^2 = |\text{OV}|^2 = 3^2+4^2 = 25$; and

length of minor axis = $2b = 6 \Rightarrow b = 3$. So, $b^2 = 9$.

Hence, the equation of the ellipse relative to the $x'y'$ -coordinate system is

$$\frac{x'^2}{25} + \frac{y'^2}{9} = 1 \qquad \text{or} \qquad 9x'^2 + 25y'^2 = 225 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

Now we use the rotation formula to express the equation relative to our xy-coordinate system. So, let θ the angle between $\boldsymbol{x}$-axis and $\boldsymbol{x}'$-axis. Then, observe that

$$\cos\theta = 4/5 \quad \text{and} \quad \sin\theta = 3/5.$$

Thus, using rotation formula (4) we get:

$$x' = \ x\cos\theta + y\sin\theta = \tfrac{4}{5}x + \tfrac{3}{5}y$$
$$y' = -x\sin\theta + y\cos\theta = \tfrac{-3}{5}x + \tfrac{4}{5}y$$

Now we substitute these for x' and y' in (1) to obtain

$$9\left(\tfrac{4}{5}x + \tfrac{3}{5}y\right)^2 + 25\left(\tfrac{-3}{5}x + \tfrac{4}{5}y\right)^2 = 225$$

And simplifying this we get

$$369x^2 - 384xy + 481y^2 - 5625 = 0$$

which is the equation of the ellipse in the xy- coordinate system.

Exercise 4.6.1

1. Suppose the ***xy***-coordinate axes are rotated 60° counterclockwise about the origin to obtain the new ***x'y'***-coordinate system.
   (a) If each of the following are coordinates of points relative to the xy-system, find the coordinates of the points relative to the ***x′y′***- system.
   (i) (5, 0) (ii) (1, 4) (iii) (0, 1) (iv) (–1/2, 5/2) (v) (–2, –1)
   (b) Find the equation of the following lines and conics relative to the new ***x′y′***- system.

   (i) $x = 5$ (iv) $(x-1)^2 + y^2 = 4$ (vii) $x^2 + 4y^2 - 4x = 0$
   (ii) $x - 2y = 1$ (v) $x^2 - 4y = 1$ (viii) $x^2 - 4y^2 = 1$
   (iii) $x^2 + y^2 = 1$ (vi) $4x^2 + (y-2)^2 = 4$ (ix) $-x^2 + y^2 - 2y = 0$

2 Suppose the ***xy***-coordinate axes are rotated 30° counterclockwise about the origin to obtain the new ***x'y'***-coordinate system. If the following points are with respect to the new ***x'y'***-system, what is the coordinates of each point with respect to the old ***xy***-system?

(a) (0, 2) (b) (–2, 4) (c) (1, –3) (d) $\left(\sqrt{3}, -\sqrt{3}\right)$

4.6.2 Analysis of the General Second Degree Equations

In the previous sections we have seen that the equation of the form

$$\text{A}x^2 + \text{C}y^2 + \text{D}x + \text{E}y + \text{F} = 0, \quad \ldots\ldots\ldots \quad (1)$$

represents a conic section (a parabola, ellipse or hyperbola) whose axis of symmetry is parallel to one of the coordinate axes except in degenerate cases. In Subsection 4.6.1 we have also seen some examples of conic sections whose equations involve xy term when their lines of symmetry are not parallel to either of the axes. Now we would like to analyze the graph of any quadratic (second degree) equation in x and y of the form

$$\text{A}x^2 + \text{B}xy + \text{C}y^2 + \text{D}x + \text{E}y + \text{F} = 0, \quad \ldots\ldots\ldots \quad (2)$$

where B≠0. In order to analyze the graph of Equation (2), we usually need to convert it into an equation of type (1) in certain suitable reference system. To this end, we first prove the following Theorem.

Theorem 4.3: Consider a general second degree equation of the form (2), i..e.,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad \text{where } B\neq 0, \quad . . . . \quad (2)$$

there is a rotation angle $\theta \in (0,\pi/2)$ through which the xy-coordinate system rotates to a new $x'y'$-coordinate system in which Equation (2) reduces to the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0. \quad . . . . \quad (3)$$

Proof: Let the xy-coordinate system rotated by an angle θ about the origin to form a new $x'y'$-coordinate system. Then, from rotation formula (3), we have

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta.$$

We can now substitute these for x and y in Equation (2) so that

$$A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\cos\theta)^2$$
$$+ D(x'\cos\theta - y'\sin\theta) + E(x'\sin\theta + y'\cos\theta) + F = 0.$$

After some calculations, combining like terms (those involving x'^2, $x'y'$, y'^2, and so on), we get equation of the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \quad . . . . \quad (4)$$

where $B' = 2(C-A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta)$.

Here the exact expressions for A', C', D', E' and F' are omitted as they are irrelevant. What we need is to get the angle of rotation θ for which Equation (4) has **no** $x'y'$ term, that is, $B' = 0$. This means that,

$$2(C-A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) = 0.$$

Since $2\sin\theta\cos\theta = \sin2\theta$ and $\cos^2\theta - \sin^2\theta = \cos2\theta$, this equation is equivalent to

$$(C-A)\sin2\theta + B\cos2\theta = 0$$

or $\dfrac{\cos2\theta}{\sin2\theta} = \dfrac{A-C}{B}$, since B≠0.

or $\cot2\theta = \dfrac{A-C}{B}$ . . . . (5).

That is, if we choose the angle of rotation θ satisfying (5), then $B' = 0$ in Equation (4) so that the resulting equation in $x'y'$-coordinate system is in the form of Equation (3). Moreover, we can always find an angle that satisfies $\cot(2\theta) = (A-C)/B$ for any $A, C, B \in \Re$, $B\neq 0$ since the range of the cotangent function is the entire set of real numbers. Note also that since $2\theta \in (0, \pi)$, the angel of rotation θ can always be chosen so that $0 < \theta < \pi/2$. So, the Theorem is proved.

Remark: If A = C, then $\cot 2\theta = \frac{A-C}{B} = 0 \;\Rightarrow\; 2\theta = \pi/2 \;\Rightarrow\; \theta = \pi/4$.

Therefore, we can rewrite the result of the above Theorem as follows:

The rotation of the xy-coordinate system by angle θ creates an $x'y'$-coordinate system in which a general second degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, $B \neq 0$,
is converted to an equation $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ (with no $x'y'$ term)
if we choose $\theta \in (0, \pi/2)$ such that

$$\tan 2\theta = \frac{B}{A - C}, \quad \text{if } A \neq C$$
$$\theta = \frac{\pi}{4}, \quad \text{if } A = C$$

Example 4.28: Use rotation of axes to eliminate the xy term in each of the following equations, describe the locus (type of conic section) and sketch the graph of the equation

a) $x^2 + 2xy + y^2 - 8\sqrt{2}x + 8\sqrt{2}y - 32 = 0$

(b) $73x^2 - 72xy + 52y^2 + 30x + 40y - 75 = 0$

Solution:

(a) Given: $x^2 + 2xy + y^2 - 8\sqrt{2}x + 8\sqrt{2}y - 32 = 0 \;\Rightarrow$ A= C=1. So, from the above Remark, the rotation angle is $\theta = \pi/4 \;\Rightarrow\; \cos\theta = \sin\theta = \frac{1}{\sqrt{2}}, \Rightarrow\; x = \frac{x' - y'}{\sqrt{2}}$ and $x = \frac{x' + y'}{\sqrt{2}}$.

Now we substitute these for x and y in the given equation:

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + \left(\frac{x'+y'}{\sqrt{2}}\right)^2 - 8\sqrt{2}\left(\frac{x'-y'}{\sqrt{2}}\right) + 8\sqrt{2}\left(\frac{x'+y'}{\sqrt{2}}\right) - 32 = 0$$

Expanding the squared expressions, combining like terms and simplifying, we obtain

$$x'^2 + 8y' - 16 = 0 \quad \text{or,} \quad x'^2 = -8(y' - 2)$$

This is an equation of a parabola. Its vertex is $(h', k') = (0,2)$ relative to the $x'y'$-system, principal axis is on y'-axis and open towards negative y' direction. (You can show that its vertex is (h, k) $= \left(-\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$ relative to the xy-system). The graph of the equation is sketched in Figure 4.34.

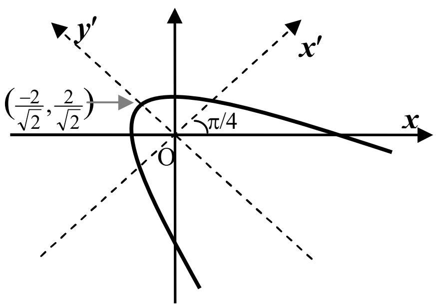


Figure 4.34: $x^2 + 2xy + y^2 - 8\sqrt{2}x + 8\sqrt{2}y - 32 = 0$

(b) Given: $73x^2 - 72xy + 52y^2 + 30x + 40y - 75 = 0 \Rightarrow$ A =73, B = −72 and C=52.
Hence,

$\tan 2\theta = \frac{B}{A-C} = -\frac{72}{21} = -\frac{24}{7} \Rightarrow$ The terminal side of 2θ is through (−7, 24) since $0 < 2\theta < \pi$.

$\Rightarrow \cos 2\theta = \frac{-7}{25}$. Now as $0 < \theta < \pi/2$, both $\cos\theta$ and $\sin\theta$ are positive. Hence,

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1-7/25}{2}} = \frac{3}{5} \quad \text{and} \quad \sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1+7/25}{2}} = \frac{4}{5}$$

This implies the x'-axis is through the coordinate point (3,4), that is the line $y = (4/3)x$.
Therefore, using the rotation formula (3), we get

$$x = \frac{3x' - 4y'}{5} \quad \text{and} \quad y = \frac{4x' + 3y'}{5}$$

Now we substitute these for x and y in the given equation to obtain

$$\tfrac{73}{25}(3x'-4y)^2 - \tfrac{72}{25}(3x'-4y')(4x'+3y') + \tfrac{52}{25}(4x'+3y')^2 + \tfrac{30}{5}(3x'-4y') + \tfrac{40}{5}(4x'+3y') - 75 = 0.$$

Expanding the squared expressions, combining like terms and simplifying, we obtain

$$25x'^2 + 100y'^2 + 50x' - 75 = 0$$

Completing the square for x' terms and divide by 100 to get

$$\frac{(x'+1)^2}{4} + y'^2 = 1$$

which is an ellipse with center at (h', k')= (−1, 0) relative to the $x'y'$-system, major axis on x'-axis (which is the line y=(4/3)x), length of major axis =4 and length of minor axis =2. (You can show that the center is $(h, k) = \left(-\frac{3}{5}, -\frac{4}{5}\right)$ relative to the xy-system). The graph of the equation is sketched in Figure 4.35.

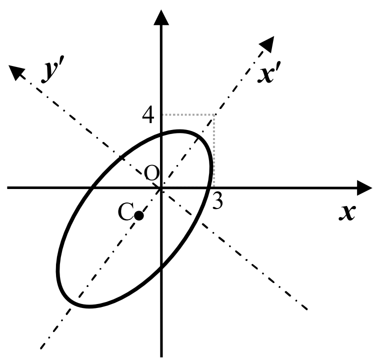


Figure 4.35: $73x^2 - 72xy + 52y^2 + 30x + 40y - 75 = 0$

Exercise 4.6.2

1. Find an equation of the conic section having the given properties and sketch its graph.
   (a) Ellipsi with center at origin, foci at (–2,2) and (2, 2), and length of major axis $2\sqrt{8}$.
   (b) Parabola whose vertex is at (3, 4) and focus (–5,–2)
   (c) Hyperbola whose foci are (–2, 2) and (2, –2), and length of transverse axis $2\sqrt{2}$.
2. Use rotation of axes to eliminate the xy term in each of the following equations, describe the locus (type of conic section) and sketch the graph of the equation.
   (a) $17x^2 - 12xy + 8y^2 - 36 = 0$
   (b) $8x^2 + 24xy + y^2 - 1 = 0$
   (c) $x^2 - 2xy + y^2 - 5y = 0$
   (d) $2x^2 + xy = 0$
   (e) $5x^2 + 6xy + 5y^2 - 4x + 4y - 4 = 0$
   (f) $x^2 + 4xy + 4y^2 + 2x - 2y + 1 = 0$
3. Show that if $B > 0$, then the graph of
   $$x^2 + Bxy = F ,$$
   is a hyperbola if $F \neq 0$, and two intersecting lines if $F = 0$.